On the motion of 3D curves and its relationship to optimal flow

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Abstract

I establish the fundamental equations that relate the three dimensional motion of a curve to its observed image motion. I introduce the notion of spatio-temporal surface and study its differential properties up to the second order. In order to do this, I only make the assumption that the 3D motion of the curve preserves arc-length, a more general assumption than that of rigid motion. I show that, contrarily to what is commonly believed, the full optical flow of the curve (i.e the component tangent to the curve) can never be recovered from this surface. I also give the equations that characterize the spatio-temporal surface completely up to a rigid transformation. Those equations are the expressions of the first and second fundamental forms and the Gauss and Codazzi-Mainardi equations.

I then show that the hypothesis of a rigid 3D motion allows in general to recover the structure and the motion of the curve, in fact without explicitely computing the tangential optical flow, at the cost of introducing the three-dimensional accelerations.

Sur le mouvement des courbes tridimensionnelles et sa relation avec le flot optique

Abstract

J'établis dans cet article les équations fondamentales reliant le mouvement d'une courbe tridimensionnelle à celui de la courbe image observée par une caméra. J'introduis pour ce faire la notion de surface spatio-temporelle et étudie ses propriétés différentielles à l'ordre deux. Cette étude est faite en supposant seulement que le mouvement tridimensionnel de la courbe préserve l'abscisse curviligne, une hypothèse plus générale que celle de mouvement rigide. Je montre que, contrairement à ce qu'on croyait, le flot optique complet (et particulièrement sa composante tangente à la courbe) ne peut pas être calculé à partir de cette surface. Je donne aussi les équations qui la caractérisent complètement modulo un déplacement rigide. Ces équations sont les coefficients des premières et deuxièmes formes quadratiques fondamentales et les équations de Gauss et de Codazzi-Mainardi.

Je montre ensuite que dans l'hypothèse d'un mouvement tridimensionnel rigide, la forme et le mouvement de la courbe 3D peuvent en général être calculés, en fait sans calculer explicitement le flot optique tangent à la courbe, à condition d'introduire les accélérations tridimensionnelles.

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1 Introduction

This article presents a mathematical formalism for dealing with the motion of curved objects, specifically curves. In our previous work on Stereo [AL87,AF89] and motion [FLT87,TDF88], we have limited ourselves to primitives such as points and lines. I attempt here to lay the ground for extending this work to general curvilinear features. More specifically, I study the image motion of 3D curves moving in a "non-elastic" way (to be defined later), such as ropes. I show that under this weak assumption the full apparent optical flow (to be defined later) can be recovered. I also show that recovering the full real optical flow (i.e the projection of the 3D velocity field) is impossible. If rigid motion is hypothesized, then I show that, in general, the full 3D structure and motion of the curve can be recovered without explicitely computing the full real flow. I assume that pixels along curves have been extracted by some standard edge detection techniques [Hil83,Can86,Der87].

This is related and inspired by the work of Koenderink and van Doorn [KvD75,KvD78, Koe86], the work of Horn and Schunk [HS81] as well as that of Longuet-Higgins and Prazdny [LP80] who pioneered the analysis of motion in computer vision, that of Nagel [Nag83] who showed first that at grey level corners the full optical flow could be recovered, as well as by the work of Hildreth [Hil84] who proposed a scheme for recovering the full flow along image intensity edges from the normal flow by using a smoothness constraint. This is also related to the work of D'Hayer [DHa86] who studied a differential equation satisfied by the optical flow but who did not relate it to the actual 3D motion and to that of Gong and Brady [Gon89] who recently extended Nagel's result and showed that it also held along intensity gradient edges. All assume, though, that the standard motion constrain equation:

$$\frac{dI}{d\tau} = \nabla I \cdot \mathbf{v} + I_\tau = 0 \tag{1}$$

is true, where I is the image intensity, $\mathbf{v}$ the optical flow (a mysterious quantity which is in fact defined by this equation). ∇I the image gradient $[I_x, I_y]^T$, and τ the time. Many techniques published in the literature use this equation to first determine the component of $\mathbf{v}$ along ∇I. Since the problem of determining the full flow is underdetermined, smoothness constraints on $\mathbf{v}$ are usually added to equation (1) to make the problem well-posed [HS81,Nag83,Hil84]. Once the optical flow has been determined, then the 3D motion is recovered through equations that are given in section 5.2 (equations (46-47)).

One problem with this approach is that it is known that equation (1) is, in general, far from being true [VP87]. In my approach, I do not make this assumption and, instead, keep explicit the relation between the image and the 3D velocities.

In fact there is a big confusion in the Computer Vision literature about the exact meaning of the optical flow. I define it precisely in this paper and show that two flows, the "apparent" and the "real" one must be distinguished. I show that only the apparent one can be recovered from the image for a large class of 3D motions.

My work is also related to that of Baker and Bolles [BB89] in the sense that I also work with spatio-temporal surfaces for which I provide a beginning of quantitative description through Differential Geometry in the case where they are generated by curves.

It is also motivated by the work of Girosi, Torre and Verri [VGT89,GVT89] who have investigated various ways of replacing equation (1) by several equations to remove the inherent ambiguity in the determination of the optical flow $\mathbf{v}$.

As a final word, the idea of computing structure and motion directly from the image without computing the optical flow is present in the work of Horn and Weldon [HW88] who propose a

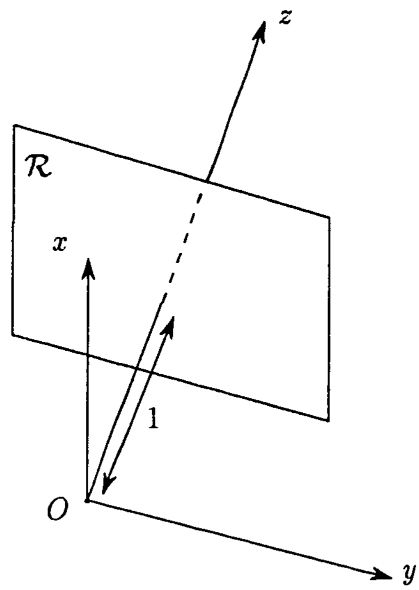

Figure 1: Geometry of image formation

solution for the case of pure translation, pure rotation, and arbitrary motion when the rotation is known and the work of Negahdaripour [NH87]. The solution I propose exploits more image structure, that is curves, and because of that does not rely upon the approximate equation (1).

2 Definitions and notations

I will make heavy use in this article of some elementary notions from Differential Geometry of curves and surfaces. I summarize these notions in the next sections and introduce my notations.

2.1 Camera model

I assume the standard pinhole model for the camera, as shown in figure 1. The retina plane $\mathcal{R}$ is perpendicular to the optical axis Oz. O is the optical center. The focal distance is assumed to be 1. Those hypothesis are quite reasonable and it is always possible, up to a good approximation, to transform a real camera into such an ideal model [Tsa86,FT86].

2.2 Two-dimensional curves

A planar curve (c) (usually in the retina plane) is defined as a C^2 mapping $u - \mathbf{m}(u)$ from an interval of R into R^2. We will assume that the parameter u is the arclength s of (c). We then have the well known two-dimensional Frenet formulas:

$$\frac{d\mathbf{m}}{ds} = \mathbf{t} \quad \frac{d\mathbf{t}}{ds} = \kappa\mathbf{n} \quad \frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t} \tag{2}$$

where $\mathbf{t}$ and $\mathbf{n}$ are the tangent and normal unit vectors to (c) at the point under consideration, and κ is the curvature of (c), the inverse of the radius of curvature r.

2.3 Three-dimensional curves

A space curve (C) is defined as a C^2 mapping $u \rightarrow \mathbf{M}(u)$ from an interval of R into R^3. We will assume that the parameter u is the arclength S of (C). We then have the well-known three-dimensional Frenet formulas:

$$\frac{d\mathbf{M}}{dS} = \mathbf{T} \quad \frac{d\mathbf{T}}{dS} = \kappa\mathbf{N} \quad \frac{d\mathbf{N}}{dS} = -\kappa\mathbf{T} - \rho\mathbf{B} \quad \frac{d\mathbf{B}}{dS} = \rho\mathbf{N} \tag{3}$$

where $\mathbf{T}$ is the tangent, $\mathbf{N}$ the normal, and $\mathbf{B}$ the binormal unit vectors to (C) at the point under consideration, κ the curvature, and ρ the torsion.

2.4 Surface patches

A surface patch (S) is defined as a C^2 mapping $(u, v) \rightarrow \mathbf{P}(u, v)$ from an open set of R^2 into R^3. Such a patch is intrinsically characterized, up to a rigid motion, by two quadratic forms, called the two fundamental forms, which are defined at every point of the patch (see section 2.4.3 for a more precise statement). The first quadratic form Φ_1 defines the length of a vector in the tangent plane T_P. More precisely, the two vectors $\mathbf{P}_u = \frac{\partial \mathbf{P}}{\partial u}$ and $\mathbf{P}_v = \frac{\partial \mathbf{P}}{\partial v}$ are parallel to this plane and define therein a system of coordinates. Each vector in the tangent plane can be defined as a linear combination $\lambda\mathbf{P}_u + \mu\mathbf{P}_v$. Its squared length is given by the value of the first fundamental form Φ_1:

$$\Phi_1(\lambda\mathbf{P}_u + \mu\mathbf{P}_v) = \lambda^2 E + 2\lambda\mu F + \mu^2 G$$

with the following definitions for E, F, G:

$$E = \|\mathbf{P}_u\|^2 \quad F = \mathbf{P}_u \cdot \mathbf{P}_v \quad G = \|\mathbf{P}_v\|^2$$

Moreover, the normal $\mathbf{N}_P$ to (S) is parallel to the cross-product $\mathbf{P}_u \times \mathbf{P}_v$ whose length is the quantity $H = \sqrt{EG - F^2}$.

The second fundamental quadratic form Φ_2 is related to curvature. For a vector $\mathbf{x} = \lambda\mathbf{P}_u + \mu\mathbf{P}_v$ in the tangent plane, we can consider all curves drawn on (S) tangent to $\mathbf{x}$ at P. These curves have all the same normal curvature, the ratio $\frac{\Phi_2(\mathbf{x})}{\Phi_1(\mathbf{x})}$, with the following definitions:

$$\Phi_2(\lambda\mathbf{P}_u + \mu\mathbf{P}_v) = \lambda^2 L + 2\lambda\mu M + \mu^2 N$$

and:

$$L = \frac{\partial^2\mathbf{P}}{\partial u^2} \cdot \frac{\mathbf{N}_P}{\|\mathbf{N}_P\|} \quad M = \frac{\partial^2\mathbf{P}}{\partial u \partial v} \cdot \frac{\mathbf{N}_P}{\|\mathbf{N}_P\|} \quad N = \frac{\partial^2\mathbf{P}}{\partial v^2} \cdot \frac{\mathbf{N}_P}{\|\mathbf{N}_P\|}$$

It is important to study the invariants of Φ_2, ie. quantities which do not depend upon the parametrization (u, v) of (S). Φ_2 defines a linear mapping $T_P \rightarrow T_P$ by $\Phi_2(\mathbf{x}) = \psi(\mathbf{x}) \cdot \mathbf{x}$. The invariants of Φ_2 are those of ψ.

2.4.1 Principal directions

The principal directions are the eigenvectors of ψ. Their coordinates (λ, μ) in the coordinate system $(\mathbf{P}_u, \mathbf{P}_v)$ are solutions of the following equation:

$$(FL - EM)\lambda^2 + (GL - EN)\lambda\mu + (GM - FN)\mu^2 = 0$$

This yields the following possible values for λ and μ (they are defined up to a scale factor):

$$\lambda = EN - GL + \epsilon\sqrt{\Delta} \tag{4}$$

$$\mu = 2(FL - EM) \tag{5}$$

where $\epsilon = \pm 1$ and $\Delta = (GL - EN)^2 - 4(FL - EM)(GM - FN)$.

2.4.2 Principal curvatures

The principal curvatures are the eigenvalues of ψ. They are solutions of the following quadratic equation:

$$(EG - F^2)\rho^2 - \rho(LG + EN - 2FM) + LN - M^2 = 0 \tag{6}$$

In particular, their product K and their half-sum H are the gaussian and mean curvatures of $(\mathcal{S})$:

$$K = \frac{LN - M^2}{EG - F^2}$$
$$H = \frac{1}{2}\frac{LG + EN - 2FM}{EG - F^2}$$

All other invariants of Φ_2 are functions of these.

2.4.3 The Gauss and Codazzi-Mainardi equations and the Bonnet theorem

We stated earlier that a surface patch was characterized, up to a rigid motion, by its first and second fundamental forms. Things are in fact a bit more subtle in the following sense. The coefficients E, F, G, L, M and N must satisfy three equations, the Gauss and Codazzi-Mainardi which we give next. Inversely, we have the Bonnet theorem:

Theorem 1 (Bonnet) *Given six differentiable functions E, F, G, L, M and N defined on an open set V of R^2 and satisfying the conditions $E > 0$, $G > 0$, $EG - F^2 > 0$, the Gauss and Codazzi-Mainardi equations. Then for every $(u, v) \in V$ there exists a diffeomorphism from a neighborhood $U \subset V$ of (u, v) in R^3 such that the corresponding regular patch has E, F, G, L, M and N as coefficients of its first and second fundamental forms. Furthermore, if U is connected, for every other such diffeomorphism, the two patches are related by a rigid transformation of R^3.*

The proof can be found in [DoC76], for example. This theorem is very important for our purpose since it shows that all the information about the spatio-temporal surface $(\mathcal{S})$ is contained in nine coefficients (the six coefficients of the first and second fundamental form and the three Gauss and Codazzi-Mainardi equations).

Those equations are obtained by expressing the derivatives of the three vectors $\mathbf{P}_u$, $\mathbf{P}_v$, and $\mathbf{N}_P$ in the basis they form and writing conditions of integrability. They can be written as [DoC76]:

$$(\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} = FK \tag{7}$$
$$L_v - M_u = L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11} \tag{8}$$
$$M_v - N_u = L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12} \tag{9}$$

where K is the gaussian curvature. The first equation is referred to as the Gauss's equation.

The coefficients Γ^k_{ij}, $i, j, k = 1, 2$ are called the Christoffel symbols of the second type. They can be easily computed from the first fundamental form, as follows. If we denote by $\mathbf{g}$ the matrix of Φ_1, the Christoffel symbols of the first kind are given by [Car88,Spi79]:

$$\Gamma_{ikj} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right) \; i, j, k = 1, 2 \tag{10}$$

They are symmetric with respect to the first and third index.

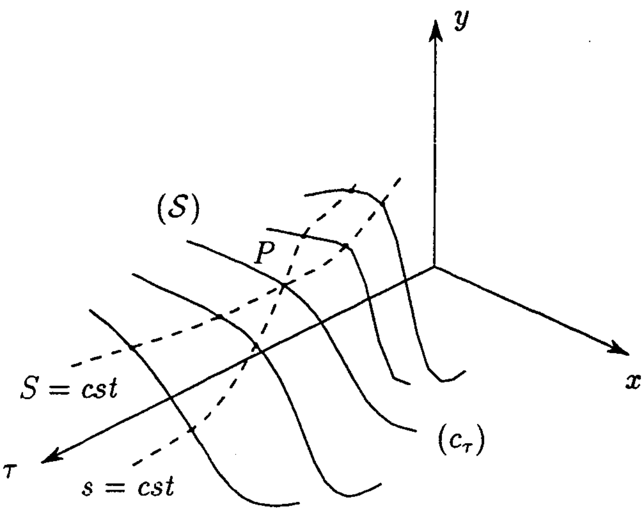

Figure 2: Definition of the spatio-temporal surface $(\mathcal{S})$

Now, following the standard tensor notation, we denote by g^{ij} the coefficients of the inverse $\mathbf{g}^{-1}$ of $\mathbf{g}$. The Christoffel symbols of the second type are then obtained as:

$$\Gamma_{ij}^{k} = g^{kh}\Gamma_{ihj} \tag{11}$$

Equations (7-11) allow us to compute the Gauss and Codazzi-Mainardi equations from the first and second fundamental forms Φ_1 and Φ_2.

3 Setting the stage: real and apparent optical flows

We now assume that we observe in a sequence of images a family (c_τ) of curves, where τ denotes the time, which we assume to be the perspective projection in the retina of a 3D curve (C) that moves in space. If we consider the three-dimensional space (x, y, τ), this family of curves sweeps in that space a surface $(\mathcal{S})$ defined as the set of points $((c_\tau), \tau)$ (see figure 2).

At a given time instant τ, let us consider the observed curve (c_τ). Its arclength s can be computed and (c_τ) can be parameterized by s and τ: it is the set of points $m_\tau(s)$. The corresponding points P on $(\mathcal{S})$ are represented by the vector $\mathbf{P} = (\mathbf{m}_\tau^T(s), \tau)^T$. The key observation is that the arclength s of (c_τ) is a function $s(S, \tau)$ of the arclength S of the 3D curve (C) and the time τ, and that the two parameters (S, τ) can be used to parameterize $(\mathcal{S})$ in a neighborhood of P. Of course, the function $s(S, \tau)$ is unknown. We cannot use (s, τ) to parameterize $(\mathcal{S})$, simply because s is itself a function of τ.

The assumption that s is a function of S and τ implies that S itself is not a function of time; in other words we do not consider here elastic motions but only motions for which S is preserved, i.e non-elastic motions such as the motion of a rope or the motion of a curve attached to a moving rigid object. We could call such motions *isometric* motions.

As shown in figure 2, we can consider on $(\mathcal{S})$ the curves defined by $s = cst$ or $S = cst$. These curves are in general different, and their projections, parallel to the τ-axis, in the (x, y)- or retina plane have an important physical interpretation, related to our upcoming definition of the optical flow.

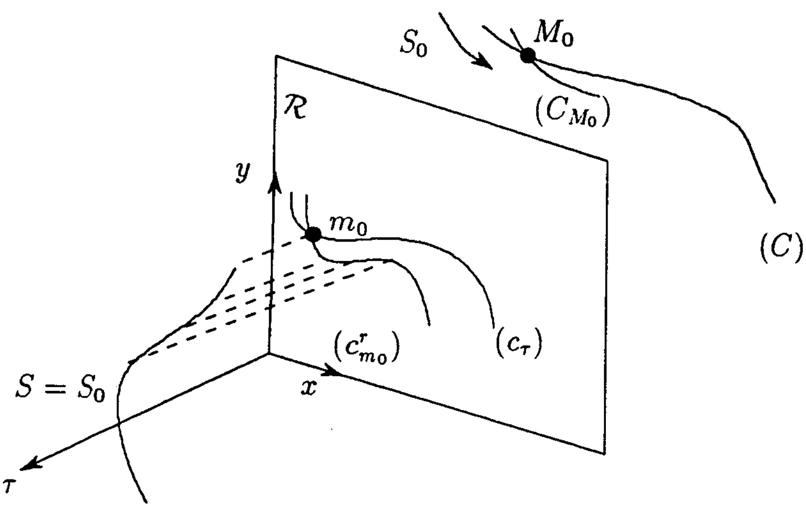

Figure 3: Projection in the image plane, parallel to the τ-axis, of the curve $S = S_0$ of the surface $(\mathcal{S})$: $(c^r_{m_0})$ is the "real" trajectory of m_0

Indeed, as shown in figure 3, suppose we choose a point M_0 on (C) and fix its arclength S_0. When (C) moves, this point follows a trajectory (C_{M_0}) in 3-space and its image m_0 follows a trajectory $(c^r_{m_0})$ in the retina plane. This last curve is the projection in the retina plane, parallel to the τ-axis, of the curve defined by $S = S_0$ on the surface $(\mathcal{S})$. We call it the "real" trajectory of m_0.

We can also consider the same projection of another curve defined on $(\mathcal{S})$ by $s = s_0$. The corresponding curve $(c^a_{m_0})$ in the retina plane is the trajectory of the image point m_0 of arclength s_0 on (c_τ). We call this curve the "apparent" trajectory of m_0 (see figure 4). The mathematical reason why those two curves are different is that the first one is defined by $S = S_0$ while the second is defined by $s(S, \tau) = s_0$.

Let me now define precisely what I mean by optical flow. If we consider figure 5, point m on (c_τ) is the image of point M on (C). This point has a $3D$ velocity $\mathbf{V}_M$ whose projection in the retina is the *real optical flow* $\mathbf{v}_r$ (r for *real*): mathematically speaking:

- $\mathbf{v}_r$ is the partial derivative of $\mathbf{m}$ with respect to time when S is kept constant, or its total time derivative.

- The *apparent optical flow* $\mathbf{v}_a$ (a for *apparent*) of m is the partial derivative with respect to time when s is kept constant.

Those two quantities are in general distinct. To relate this to the previous discussion about the curves $S = S_0$ and $s = s_0$ of $(\mathcal{S})$, the vector $\mathbf{v}_a$ is tangent to the "apparent" trajectory of m, while $\mathbf{v}_r$ is tangent to the "real" one. This is summarized in figure 6.

I now make the following fundamental remark. All the information about the motion of points of (c_τ) (and of the 3D points of (C) which project onto them) is entirely contained in the surface $(\mathcal{S})$. Since $(\mathcal{S})$ is intrinsically characterized, up to a rigid motion, by its first and second fundamental forms and the Gauss and Codazzi-Mainardi equations [DoC76], they are all we need to characterize the optical flow of (c_τ) and the motion of (C).

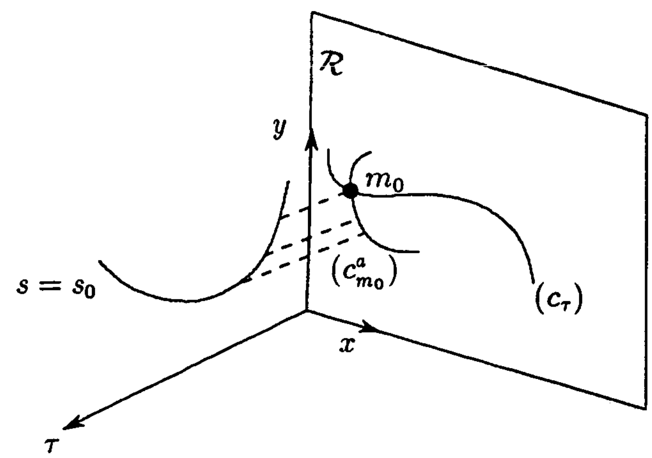

Figure 4: Projection in the image plane, parallel to the τ-axis, of the curve $s = s_0$ of the surface $(\mathcal{S})$: $(c^a_{m_0})$ is the "apparent" trajectory of m_0

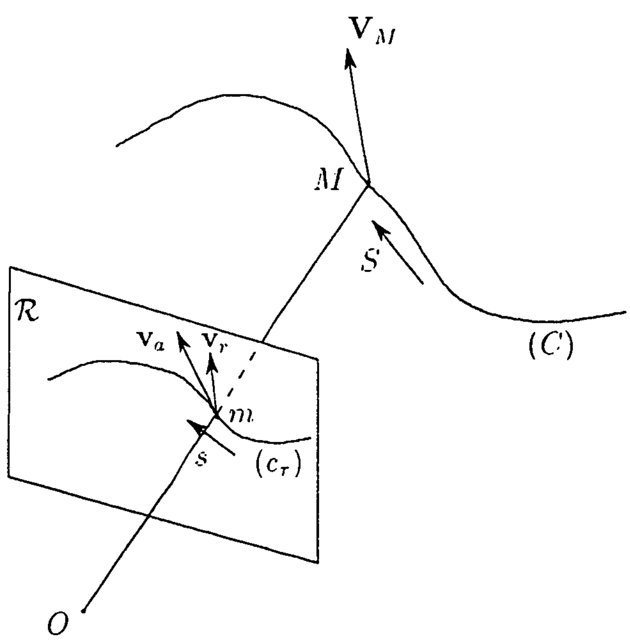

Figure 5: Definition of the two optical flows: the real and the apparent

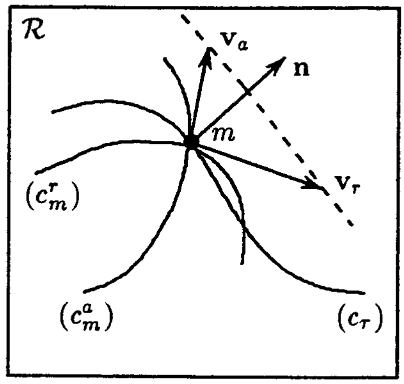

Figure 6: Comparison of the two optical flows and the real and apparent trajectories: $\mathbf{n}$ is the normal to (c_τ)

4 Characterization of the spatio-temporal surface $(\mathcal{S})$

In this section, we compute the first and second fundamental forms and the Codazzi-Mainardi equations of the spatio-temporal surface $(\mathcal{S})$. We will be using over and over again the following result.

Given a function f of the variables s and τ, it is also a function f' of S and τ. We will have to compute $\frac{\partial f'}{\partial S}$ and $\frac{\partial f'}{\partial \tau}$, also called the total time derivative of f with respect to time, $\dot{f}$; introducing $u = \frac{\partial s}{\partial S}$ and $v = \frac{\partial s}{\partial \tau}$, we have the following equations:

$$\frac{\partial f'}{\partial S} = u\frac{\partial f}{\partial s} \quad \dot{f} = \frac{\partial f'}{\partial \tau} = v\frac{\partial f}{\partial s} + \frac{\partial f}{\partial \tau} \tag{12}$$

Following these notations, we denote by $\mathbf{P}(s,\tau) = (\mathbf{m}^T(s,\tau),\tau)^T$ the generic point of $(\mathcal{S})$ and by $\mathbf{P}'(S,\tau) = (\mathbf{m}'^T(S,\tau),\tau)^T$ the same point considered as a function of S and τ.

The second equation (12) giving $\dot{f}$ is not very satisfying because $v = \frac{\partial s}{\partial \tau}$ does not have any intuitive interpretation and $\frac{\partial f}{\partial \tau}$ is difficult to compute from the observation of (c_τ) or of $(\mathcal{S})$. In the next section, we give more satisfying equations for $\frac{\partial f}{\partial \tau}$ and $\dot{f}$.

4.1 Computation of the first fundamental form

Using equations (12), we write immediatly:

$$\mathbf{P}'_S = u\mathbf{P}_s = \begin{bmatrix} u\mathbf{t} \\ 0 \end{bmatrix} \tag{13}$$

$$\mathbf{P}'_\tau = v\mathbf{P}_s + \mathbf{P}_\tau = \begin{bmatrix} v\mathbf{t} + \mathbf{v}_a \\ 1 \end{bmatrix} \tag{14}$$

Let $\mathbf{V}_a = [\mathbf{v}_a^T, 1]^T$. We now write the apparent optical flow $\mathbf{v}_a$ in the reference frame defined by $\mathbf{t}$ and $\mathbf{n}$:

$$\mathbf{v}_a = \alpha\mathbf{t} + \beta\mathbf{n} \tag{15}$$

We see from equation (14) that $\mathbf{P}'_\tau = [(v+\alpha)\mathbf{t}^T + \beta\mathbf{n}^T, 1]^T$; but by definition, $\mathbf{P}'_\tau = [\mathbf{v}_\tau^T, 1]^T$. Therefore $w = v + \alpha$ is the real tangential optical flow and β the normal real optical flow.

Therefore, the real and apparent optical flows have the same component along $\mathbf{n}$, we call it the normal optical flow (see figure 6). The real optical flow is given by:

$$\mathbf{v}_r = w\mathbf{t} + \beta\mathbf{n} \tag{16}$$

We also define $\mathbf{V}_r$ as $[\mathbf{v}_r^T, 1]^T$.

A further remark of interest is that the tangent plane T_P to $(\mathcal{S})$ at P is spanned by the two vectors $\mathbf{P}'_S$ and $\mathbf{P}'_\tau$, by definition. Examining those two vectors, we see that the vectors $\mathbf{t}_0 = [\mathbf{t}^T, 0]^T$ and $\mathbf{n}_\beta = [\beta\mathbf{n}^T, 1]^T$ which are orthogonal, also span T_P. It should be clear that the two vectors $\mathbf{V}_r = w\mathbf{t}_0 + \mathbf{n}_\beta$ and $\mathbf{V}_a = \alpha\mathbf{t}_0 + \mathbf{n}_\beta$ belong to T_P and define on $(\mathcal{S})$ two tangent vector fields.

Expanding on this idea, we can give a geometric interpretation of the operation of partial derivative $\frac{\partial}{\partial\tau}$ when the image arclength s is kept constant. Given a function f from $(\mathcal{S})$ into R, for example, $\frac{\partial f}{\partial\tau}$ is the Lie derivative $L_{\mathbf{V}_a}f$ of the function f with respect to the tangent vector field $\mathbf{V}_a$ defined on $(\mathcal{S})$. Similarly, the total time derivative $\dot{f}$ is the Lie derivative $L_{\mathbf{V}_r}f$ of f with respect to the tangent vector field $\mathbf{V}_r$. In particular, if we denote by Df, the derivative of f at point P on $(\mathcal{S})$ (a linear mapping from R^3 to R), we have:

$$\frac{\partial f}{\partial\tau} = L_{\mathbf{V}_a}f = \alpha Df(\mathbf{t}_0) + Df(\mathbf{n}_\beta)$$
$$\dot{f} = L_{\mathbf{V}_r}f = wDf(\mathbf{t}_0) + Df(\mathbf{n}_\beta)$$

Noticing that $Df(\mathbf{t}_0) = \frac{\partial f}{\partial s}$ and denoting $Df(\mathbf{n}_\beta)$ by $\partial_{\mathbf{n}_\beta}f$, meaning the partial derivative of f in the direction of the vector $\mathbf{n}_\beta$ of T_P, we have:

$$\frac{\partial f}{\partial\tau} = \alpha\frac{\partial f}{\partial s} + \partial_{\mathbf{n}_\beta}f \tag{17}$$

$$\dot{f} = w\frac{\partial f}{\partial s} + \partial_{\mathbf{n}_\beta}f \tag{18}$$

These relations also hold for functions f from $(\mathcal{S})$ into R^p. We will be using heavily the cases $p = 2, 3$ in what follows.

From equations (13) and (14), we can compute the coefficients of the first fundamental form:

Proposition 1 *The coefficients of the first fundamental form in the basis* $(\mathbf{P}'_S, \mathbf{P}'_\tau)$ *of* T_P *are given by:*

$$E = u^2 \quad F = uw \quad G = 1 + w^2 + \beta^2 \tag{19}$$

We can also compute those coefficients in the basis $(\mathbf{t}_0, \mathbf{n}_\beta)$:

Proposition 2 *The coefficients of the first fundamental form in the basis* $(\mathbf{t}_0, \ \mathbf{n}_\beta)$ *are given by:*

$$E' = 1 \quad F' = 0 \quad G' = 1 + \beta^2$$

Proof : Let us denote φ the linear mapping $T_P \to T_P$ such that $\Phi_1\mathbf{x} = \varphi\mathbf{x}\cdot\mathbf{x}$ for all $\mathbf{x}$ of T_P. Since we have:

$$\mathbf{t}_0 = \tfrac{1}{u}\mathbf{P}'_S \quad \mathbf{n}_\beta = -\tfrac{w}{u}\mathbf{P}'_S + \mathbf{P}'_\tau$$

we have immediately $E' = \Phi_1\mathbf{t}_0 = \frac{1}{u^2}\Phi_1\mathbf{P}'_S = \frac{E}{u^2} = 1$, $G' = \Phi_1\mathbf{n}_\beta = \frac{w^2}{u^2}E - 2\frac{w}{u}F + G = 1 + \beta^2$, and $F' = \varphi\mathbf{t}_0\cdot\mathbf{n}_\beta = -\frac{w}{u^2}E + \frac{1}{u}F = 0$. □

We can also compute the normal $\mathbf{N}_P$ which will be needed for the second fundamental form.

$$\mathbf{N}_P = \mathbf{P}'_S \times \mathbf{P}'_\tau = uv \begin{bmatrix} \mathbf{t} \\ 0 \end{bmatrix} \times \begin{bmatrix} \mathbf{t} \\ 0 \end{bmatrix} + u \begin{bmatrix} \mathbf{t} \\ 0 \end{bmatrix} \times \begin{bmatrix} \mathbf{v}_a \\ 1 \end{bmatrix} = u \begin{bmatrix} t_y \\ -t_x \\ \mathbf{t} \times \mathbf{v}_a \end{bmatrix} = u \begin{bmatrix} \epsilon\mathbf{n} \\ \mathbf{t} \times \mathbf{v}_a \end{bmatrix}$$

where $\epsilon = \pm 1$. In this equation, the cross-product $\mathbf{t} \times \mathbf{v}_a$ is to be taken as the algebraic value of the corresponding vector which is parallel to the τ-axis. From the definition of $\mathbf{v}_a$, this value is equal to $-\epsilon\beta$. Thus we have:

$$\mathbf{N}_P = \epsilon u \begin{bmatrix} \mathbf{n} \\ -\beta \end{bmatrix}$$

Now, given the normal $\mathbf{N}_P$ to the spatio-temporal surface($\mathcal{S}$) whose coordinates in the coordinate system $(\mathbf{t}, \mathbf{n}, \tau)$ (τ is the unit vector defining the τ-axis) are denoted by N_1, N_2, N_3, we deduce:

$$\beta = -\frac{N_3}{N_2} \quad N_1 = 0$$

We have thus proved the following proposition:

Proposition 3 *The normal to the spatio-temporal surface* ($\mathcal{S}$) *yields an estimate of the* normal optical flow β.

In what follows we take $\mathbf{N}_P = \begin{bmatrix} \mathbf{n} \\ -\beta \end{bmatrix}$ since $\mathbf{N}_P$ is defined up to a scale factor.

4.2 Computation of the second fundamental form

We denote $\frac{\partial^2 s}{\partial S^2}$ by u_S, $\frac{\partial^2 s}{\partial \tau^2}$ by v_τ, $\frac{\partial^2 s}{\partial \tau \partial S}$ by u_τ, and $\frac{\partial^2 s}{\partial S \partial \tau}$ by v_S. We are going to compute $\frac{\partial^2 \mathbf{P}'}{\partial S^2}$, $\frac{\partial^2 \mathbf{P}'}{\partial \tau^2}$, and $\frac{\partial^2 \mathbf{P}'}{\partial S \partial \tau}$. We start with the last one; using equation (14) and the first equation (12), we deduce:

$$\frac{\partial^2 \mathbf{P}'}{\partial S \partial \tau} = \begin{bmatrix} v_S \mathbf{t} + u(v\kappa\mathbf{n} + \frac{\partial \mathbf{v}_a}{\partial s}) \\ 0 \end{bmatrix}$$

Let us now evaluate $\frac{\partial \mathbf{v}_a}{\partial s}$. From the definition of $\mathbf{v}_a$ (equation (15)) and the two-dimensional Frenet formulas (2), we infer:

$$\frac{\partial \mathbf{v}_a}{\partial s} = (\frac{\partial \alpha}{\partial s} - \kappa\beta)\mathbf{t} + (\kappa\alpha + \frac{\partial \beta}{\partial s})\mathbf{n} \tag{20}$$

Thus:

$$\frac{\partial^2 \mathbf{P}'}{\partial S \partial \tau} = \begin{bmatrix} (v_S + u(\frac{\partial \alpha}{\partial s} - \kappa\beta))\mathbf{t} + u(\kappa w + \frac{\partial \beta}{\partial s})\mathbf{n} \\ 0 \end{bmatrix} \tag{21}$$

Let us now compute $\frac{\partial^2 \mathbf{P}'}{\partial \tau \partial S}$; from equation (13), we deduce:

$$\frac{\partial^2 \mathbf{P}'}{\partial \tau \partial S} = \begin{bmatrix} u_\tau \mathbf{t} + u\dot{\mathbf{t}} \\ 0 \end{bmatrix} \tag{22}$$

Using the derivation rule (18), we write:

$$\dot{\mathbf{t}} = w\frac{\partial \mathbf{t}}{\partial s} + \partial_{\mathbf{n}_\beta}\mathbf{t} \tag{23}$$

Now, the vector $\partial_{\mathbf{n}_\beta}\mathbf{t}$ is a derivative of the unit vector $\mathbf{t}$; It is therefore perpendicular to $\mathbf{t}$, thus parallel to $\mathbf{n}$. We write:

$$\partial_{\mathbf{n}_\beta}\mathbf{t} = \varphi\mathbf{n} \tag{24}$$

We call φ the β-curvature of (c_τ) while κ is the space curvature. Of course, we also have:

$$\partial_{\mathbf{n}_\beta}\mathbf{n} = -\varphi\mathbf{t} \tag{25}$$

Therefore, equation (23) yields:

$$\dot{\mathbf{t}} = (\kappa w + \varphi)\mathbf{n} \tag{26}$$

From Schwarz equality $\frac{\partial^2 \mathbf{P}'}{\partial S \partial \tau} = \frac{\partial^2 \mathbf{P}'}{\partial \tau \partial S}$, and $u_\tau = v_S$; from this, we conclude, by equating equations (21) and (22) that, if $u \neq 0$, we have the following theorem:

Theorem 2 *The tangential apparent optical flow α and the β-curvature φ satisfy:*

$$\frac{\partial \alpha}{\partial s} = \kappa\beta \tag{27}$$

$$\varphi = \frac{\partial \beta}{\partial s} \tag{28}$$

Equation (27) is instructive. Indeed, it shows that α, the tangential component of the apparent optical flow $\mathbf{v}_a$ is entirely determined up to the addition of a function of time by the normal component of the optical flow β and the space curvature κ of (c_τ):

$$\alpha = \int_{s_0}^{s} \kappa(t, \tau)\beta(t, \tau)dt \tag{29}$$

Changing the origin of arclengths from s_0 to s_1 on (c_τ) is equivalent to adding the function $\int_{s_0}^{s_1} \kappa(t, \tau)\beta(t, \tau)dt$ to α, function which is constant on (c_τ). This is the fundamental result of this section. We have proved the following theorem:

Theorem 3 *The tangential apparent optical flow can be recovered from the normal flow up to the addition of a function of time through equation (29).*

The choice of this function of time is related to the choice of the origin of arclengths at each time instant. In practice, we want this function of time to be smooth, i.e if $Or(\tau)$ is the origin at time τ, its tangential apparent velocity is 0 by definition (equation (29)). and we would like the origin at time $Or(\tau + d\tau)$ to be $Or(\tau) + \beta\mathbf{n}d\tau$ (see figure 7). Mathematically, this means that if P_{or} is the corresponding point on (S), it follows a trajectory described by the following differential equation[1]:

$$\frac{dP_{or}}{d\tau}(\tau) = \mathbf{n}_\beta(\tau) \tag{30}$$

We know from the theory of differential equations that if $\mathbf{n}_\beta$ is smooth enough, then equation (30) has a unique solution for the initial condition $P_{or}(0) = P_0$ in a neighbourhood of P_0. Thus, since $\mathbf{n}_\beta(\tau)$ is known at each time instant, if we choose the origin as P_0 at time 0, we can compute what the origin is at time τ. and compute α from equation (29).

We now prove an interesting relationship between the β- and space curvatures of (c_τ).

[1]Or equivalently, the point Or in the image follows a trajectory defined by: $\frac{dOr}{d\tau}(\tau) = \beta\mathbf{n}$

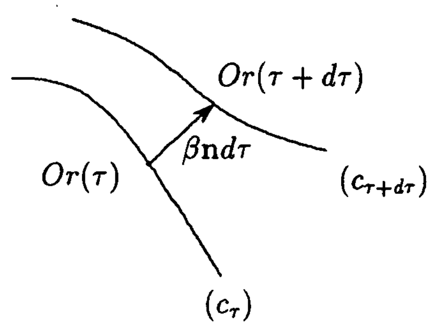

Figure 7: The choice of the origin of arclength on (c_τ)

Proposition 4 *The β- and space curvatures of (c_τ) satisfy:*

$$\frac{\partial \varphi}{\partial s} = \partial_{\mathbf{n}_\beta}\kappa - \kappa^2 \beta$$

which can be rewritten as:

$$\partial_{\mathbf{n}_\beta}\kappa = \frac{\partial^2 \beta}{\partial s^2} + \kappa^2 \beta \tag{31}$$

Proof : We compute $\frac{\partial^2 \mathbf{t}}{\partial s \partial \tau}$ and $\frac{\partial^2 \mathbf{t}}{\partial \tau \partial s}$ and write that they are equal. Using the derivation rule (17), we can write:

$$\frac{\partial^2 \mathbf{t}}{\partial s \partial \tau} = \frac{\partial(\alpha \frac{\partial \mathbf{t}}{\partial s} + \partial_{\mathbf{n}_\beta}\mathbf{t})}{\partial s} = \frac{\partial((\kappa\alpha + \varphi)\mathbf{n})}{\partial s}$$

thus:

$$\frac{\partial^2 \mathbf{t}}{\partial s \partial \tau} = (\frac{\partial \kappa}{\partial s}\alpha + \kappa\frac{\partial \alpha}{\partial s} + \frac{\partial \varphi}{\partial s})\mathbf{n} - \kappa(\kappa\alpha + \varphi)\mathbf{t}$$

On the other hand:

$$\frac{\partial^2 \mathbf{t}}{\partial \tau \partial s} = \frac{\partial(\kappa \mathbf{n})}{\partial \tau} = \alpha\frac{\partial(\kappa \mathbf{n})}{\partial s} + \partial_{\mathbf{n}_\beta}(\kappa \mathbf{n})$$

thus:

$$\frac{\partial^2 \mathbf{t}}{\partial \tau \partial s} = (\alpha\frac{\partial \kappa}{\partial s} + \partial_{\mathbf{n}_\beta}\kappa)\mathbf{n} - \kappa(\kappa\alpha + \varphi)\mathbf{t}$$

From where the announced results follow. □

Just like the previous result has been obtained by writing that $\frac{\partial^2 \mathbf{t}}{\partial s \partial \tau} = \frac{\partial^2 \mathbf{t}}{\partial \tau \partial s}$, we can obtain another result by applying the same technique to the other fundamental quantity of our problem, namely β. We thus obtain the proposition:

Proposition 5 *The normal flow, β- and spatial curvatures satisfy the following relationship:*

$$\partial_{\mathbf{n}_\beta}(\frac{\partial \beta}{\partial s}) - \frac{\partial(\partial_{\mathbf{n}_\beta}\beta)}{\partial s} = \kappa\beta\frac{\partial \beta}{\partial s} \tag{32}$$

Proof : The proof is straightforward and uses the derivation rule (17). □

There is an interesting interpretation of equation (32) in terms of Lie brackets. Indeed, its left-hand side is recognized as the Lie derivative $L_{[\mathbf{n}_\beta, \mathbf{t}_0]}\beta$ of β with respect to the Lie bracket

$[\mathbf{n}_\beta, \mathbf{t}_0]$. It is easy to show (see Appendix A) that $L_{[\mathbf{n}_\beta, \mathbf{t}_0]}\beta = \frac{\partial\alpha}{\partial s}\frac{\partial\beta}{\partial s}$. Therefore, equation (32) can be rewritten as:

$$\frac{\partial\alpha}{\partial s}\frac{\partial\beta}{\partial s} = \kappa\beta\frac{\partial\beta}{\partial s}$$

and, if $\frac{\partial\beta}{\partial s} \neq 0$, we find equation (27) again. Equation (32) is not "new", it is equivalent to equation (27) in the case where $\frac{\partial\beta}{\partial s} \neq 0$.

Let us now evaluate $\frac{\partial^2\mathbf{P}'}{\partial S^2}$ and $\frac{\partial^2\mathbf{P}'}{\partial\tau^2}$; from equation (13) and the first equation of (12) we derive:

$$\mathbf{P}'_{S^2} = u^2\mathbf{P}_{s^2} + u_S\mathbf{P}_s = \begin{bmatrix} u^2\kappa\mathbf{n} + u_S\mathbf{t} \\ 0 \end{bmatrix}$$

and from equation (14):

$$\mathbf{P}'_{\tau^2} = \begin{bmatrix} v_\tau\mathbf{t} + v\dot{\mathbf{t}} + \dot{\mathbf{v}}_a \\ 0 \end{bmatrix}$$

We have seen previously that $\dot{\mathbf{t}} = (\kappa w + \varphi)\mathbf{n}$ (equation (26)); now, using again the derivation rule (18), we obtain:

$$\dot{\mathbf{v}}_a = w\frac{\partial\mathbf{v}_a}{\partial s} + \partial_{\mathbf{n}_\beta}\mathbf{v}_a$$

we know from equations (20) and (27) that $\frac{\partial\mathbf{v}_a}{\partial s} = (\kappa\alpha + \varphi)\mathbf{n}$, we evaluate $\partial_{\mathbf{n}_\beta}\mathbf{v}_a$:

$$\partial_{\mathbf{n}_\beta}\mathbf{v}_a = (\partial_{\mathbf{n}_\beta}\alpha - \beta\varphi)\mathbf{t} + (\alpha\varphi + \partial_{\mathbf{n}_\beta}\beta)\mathbf{n}$$

so that, finally:

$$\mathbf{P}'_{\tau^2} = \begin{bmatrix} (v_\tau + \partial_{\mathbf{n}_\beta}\alpha - \beta\varphi)\mathbf{t} + (\kappa w^2 + 2w\varphi + \partial_{\mathbf{n}_\beta}\beta)\mathbf{n} \\ 0 \end{bmatrix}$$

We can now compute the coefficients of the second fundamental form; after some algebra, and using equation (28), we obtain:

Proposition 6 *The coefficients of the second fundamental form in the basis* $(\mathbf{P}'_S, \mathbf{P}'_\tau)$ *of* T_P *are given by:*

$$L = \frac{\kappa u^2}{\sqrt{1+\beta^2}} \quad M = \frac{(\kappa w + \frac{\partial\beta}{\partial s})u}{\sqrt{1+\beta^2}} \quad N = \frac{\kappa w^2 + 2w\frac{\partial\beta}{\partial s} + \partial_{\mathbf{n}_\beta}\beta}{\sqrt{1+\beta^2}} \tag{33}$$

We can also compute those coefficients in the basis $(\mathbf{t}_0, \mathbf{n}_\beta$:

Proposition 7 *The coefficients of the second fundamental form in the basis* $(\mathbf{t}_0, \mathbf{n}_\beta)$ *are given by:*

$$L' = \frac{\kappa}{\sqrt{1+\beta^2}} \quad M' = \frac{\frac{\partial\beta}{\partial s}}{\sqrt{1+\beta^2}} \quad N' = \frac{\partial_{\mathbf{n}_\beta}\beta}{\sqrt{1+\beta^2}}$$

Proof : Let us denote ψ the linear mapping $T_P - T_P$ such that $\Phi_2\mathbf{x} = \psi\mathbf{x}\cdot\mathbf{x}$ for all $\mathbf{x}$ of T_P. Since we have:

$$\mathbf{t}_0 = \tfrac{1}{u}\mathbf{P}'_S \quad \mathbf{n}_\beta = -\tfrac{w}{u}\mathbf{P}'_S + \mathbf{P}'_\tau$$

we can write $L' = \Phi_2\mathbf{t}_0 = \frac{1}{u^2}\Phi_2\mathbf{P}'_S = \frac{L}{u^2}$, $N' = \Phi_2\mathbf{n}_\beta = \frac{w^2}{u^2}L - 2\frac{w}{u}M + N = \frac{\partial_{\mathbf{n}_\beta}\beta}{\sqrt{1+\beta^2}}$, and $M' = \psi\mathbf{t}_0\cdot\mathbf{n}_\beta = -\frac{w}{u^2}L + \frac{1}{u}\psi\mathbf{P}'_S\cdot\mathbf{P}'_\tau = -\frac{w}{u^2}L + \frac{1}{u}M = \frac{\frac{\partial\beta}{\partial s}}{\sqrt{1+\beta^2}}$ □

4.3 What information can be extracted from the second fundamental form

The idea now is that after observing $(\mathcal{S})$, we compute an estimate of Φ_2 (see section 7) from which we attempt to recover the unknowns, for example u or v. We show that it is impossible without making stronger assumptions about the motion of (C).

We have seen that all invariants of Φ_2 are functions of the principal directions and curvatures.

Let us look at the principal curvatures first, and compute the gaussian and mean curvatures of $(\mathcal{S})$. Some simple algebra yields:

$$K = \frac{\kappa\partial_{\mathbf{n}_\beta}\beta - (\frac{\partial\beta}{\partial s})^2}{(1+\beta^2)^2}$$

$$H = \frac{1}{2}\,\frac{\kappa(1+\beta^2) + \partial_{\mathbf{n}_\beta}\beta}{(1+\beta^2)^{\frac{3}{2}}}$$

Those two equations show that any function of the principal curvatures is not a function of u or v.

Let us now look at the principal directions. First of all, how many new equations can we hope to obtain from them ? they are two orthogonal directions in a known plane T_P. Therefore, once one direction is known, the other one is also known, and we obtain only one equation.

From the results of section 2.4, we can compute the coordinates λ and μ of the principal directions in the tangent plane, let us denote one of them by $\mathbf{v}_p$. We have:

$$\mathbf{v}_p = \lambda\mathbf{P}'_S + \mu\mathbf{P}'_\tau = \begin{bmatrix} (\lambda u + \mu w)\mathbf{t} + \mu\beta\mathbf{n} \\ \mu \end{bmatrix}$$

Now, we can *estimate* such a direction $\hat{\mathbf{v}}_p$ from the observation of $(\mathcal{S})$. Let us denote by M_1, M_2, M_3 its coordinates in the system $\mathbf{t}$, $\mathbf{n}$, τ, we thus have the two equalities:

$$\frac{\lambda u + \mu w}{M_1} = \frac{\mu\beta}{M_2} = \frac{\mu}{M_3}$$

from which we deduce the following equation:

$$\lambda u M_3 = \mu(M_1 - M_3 w)$$

Replacing λ and μ by their values obtained from equation (5), we obtain:

$$-uM_3(EN - GL) + 2(FL - EM)(M_1 - M_3 w) = uM_3\epsilon\sqrt{\Delta}$$

Taking the square of both sides and replacing Δ by its value, we can write, if $FL - EM \neq 0$:

$$uM_3(M_1 - M_3 w)(EN - GL) - u^2 M_3^2(GM - FN) = (FL - EM)(M_1 - M_3 w)^2$$

The values of $FL - EM$, $EN - GL$, and $GM - FN$ are given by:

$$FL - EM = -\frac{u^3\frac{\partial\beta}{\partial s}}{(1+\beta^2)^{\frac{1}{2}}}$$

$$GL - EN = \frac{u^2(\kappa(1+\beta^2) - 2w\frac{\partial\beta}{\partial s} - \partial_{\mathbf{n}_\beta}\beta)}{(1+\beta^2)^{\frac{1}{2}}}$$

$$GM - FN = \frac{u((1+\beta^2)(\kappa w + \frac{\partial\beta}{\partial s}) - w(w\frac{\partial\beta}{\partial s} + \partial_{\mathbf{n}_\beta}\beta))}{(1+\beta^2)^{\frac{1}{2}}}$$

This shows that the condition $FL - EM \neq 0$ is equivalent to $\frac{\partial \beta}{\partial s} \neq 0$; we thus obtain the announced equation (assuming $u \neq 0$):

$$M_1 M_3 \partial_{\mathbf{n}_\beta} \beta + \kappa M_1 M_3 (1+\beta^2) + \frac{\partial \beta}{\partial s}(M_3^2(1+\beta^2) - M_1^2) = 0$$

which shows that the principal directions do not yield either any information about u or v. We have therefore proved the following proposition:

Proposition 8 *The invariants of the second fundamental form of the surface $(\mathcal{S})$ are not functions of u, v, w, the real tangential optical flow nor of α, the apparent tangential optical flow.*

4.4 Did we get them all ?

The reader may wonder whether we have completely characterized the spatio-temporal surface $(\mathcal{S})$ and if it is not possible to find other relations that may yield more information than what we have found so far. The answer to this is no, thanks to the Bonnet theorem which we quoted in section (2.4). Indeed, it can be shown by combining equations (7-11) with the expressions (19) and (33) that the Gauss and Codazzi-Mainardi equations (7, 8, 9) for $(\mathcal{S})$ imply equations (31), (27), and (32). Indeed, it can be shown that the Gauss and Codazzi-Mainardi equations can be written as:

$$w u \beta (\kappa^2 \beta + \frac{\partial^2 \beta}{\partial s^2} - \partial_{\mathbf{n}_\beta} \kappa)/(1+\beta^2) = 0$$

$$u(\kappa(1+\beta^2) - \partial_{\mathbf{n}_\beta}\beta)(u_\tau - w_S + \kappa u \beta)/(1+\beta^2)^{3/2} = 0$$

$$u(1+\beta^2)(\kappa\beta\frac{\partial \beta}{\partial s} - \partial_{\mathbf{n}_\beta}(\frac{\partial \beta}{\partial s}) + \frac{\partial(\partial_{\mathbf{n}_\beta}\beta)}{\partial s})/(1+\beta^2)^{3/2} = 0$$

The first equation implies equation (31), the third one implies equation (32). But we have seen that equation (32) also implied equation (27). Now, using the facts that $w = v + \alpha$, $v_S = u_\tau$, and $\alpha_S = u\frac{\partial \alpha}{\partial s}$, we can write the second equation as:

$$u^2((\kappa(1+\beta^2) - \partial_{\mathbf{n}_\beta}\beta)(\frac{\partial \alpha}{\partial s} - \kappa\beta)/(1+\beta^2)^{3/2} = 0$$

Therefore it appears that this equation is a consequence of the third and that the condition $\partial_{\mathbf{n}_\beta}\beta = \kappa(1+\beta^2)$ is not necessarily true. We will show it again later in the simple case of the retinal rigid motion (see section 6).

From the Bonnet theorem, the two equations (27) and (31), together with those giving the coefficients of the first and second fundamental forms (equations (19) and (33)), completely characterize $(\mathcal{S})$, up to a rigid motion.

4.5 Conclusions

There are three main consequences that we can draw from this analysis. Under the weak assumption of *isometric* motion:

1. The normal optical flow β can be recovered from the normal to the spatio-temporal surface,
2. The tangential apparent optical flow can be recovered from the normal optical flow through equation (29), up to the addition of a function of time, and we have seen how to eliminate this problem,

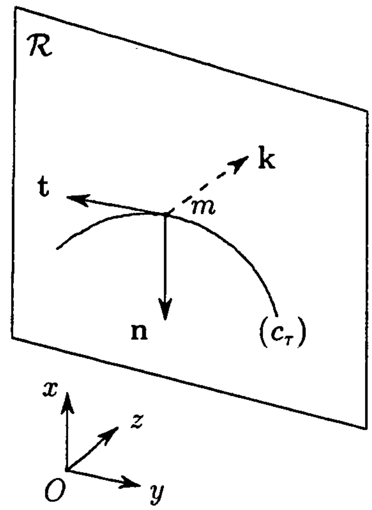

Figure 8: Definition of the local system of coordinates

3. The tangential real optical flow cannot be recovered from the spatio-temporal surface.

Therefore, the full real optical flow is not computable from the observation of the image of a moving curve under the isometric assumption. In order to compute it we must add more hypothesis, for example that the 3D motion is rigid.

I show in the next section that if we assume a 3D rigid motion then the problem is, in general, solvable but that there is no need to compute the full real optical flow.

5 Assuming that (C') is moving rigidly

We are now assuming that (C') is moving rigidly; let $(\mathbf{\Omega}, \mathbf{V})$ be its kinematic screw at the optical center O of the camera. We first derive a fundamental relation between the tangents $\mathbf{t}$ and $\mathbf{T}$ to (c_τ) and (C') and the angular velocity $\mathbf{\Omega}$. We will be using constantly the coordinate system $(\mathbf{t}, \mathbf{n}, \mathbf{k})$, where $\mathbf{k}$ is the unit vector along the z-axis (see figure 8). In this section, the third coordinate of vectors is a space coordinate (along the z-axis) whereas previously it was a time coordinate (along the τ-axis).

5.1 Stories of tangents

In this section, as stated in section 4, the $\dot{}$ indicates total time derivative, ie. derivative with respect to time, S being kept constant. Let us denote by $\mathbf{U_t}$ the vector $\mathbf{Om} \times \begin{bmatrix} \mathbf{t} \\ 0 \end{bmatrix}$. This vector is normal to the plane defined by the optical center of the camera, the point m on (c_τ), and $\mathbf{t}$ (see figure 9). Since this plane contains also the tangent $\mathbf{T}$ to (C') at M, the 3D point whose image is m, we have:

$$\mathbf{U_t} \cdot \mathbf{T} = 0 \tag{34}$$

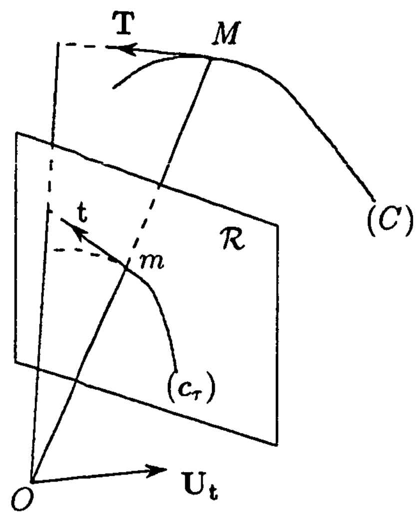

Figure 9: Relation between **t** and **T**

But, because (C) moves rigidly, **T** must also satisfy the following differential equation[2]:

$$\dot{\mathbf{T}} = \boldsymbol{\Omega} \times \mathbf{T} \tag{35}$$

Taking the time derivative of equation (34), we readily obtain:

$$\dot{\mathbf{U}}_\mathbf{t} \cdot \mathbf{T} + \mathbf{U_t} \cdot \dot{\mathbf{T}} = 0 \tag{36}$$

Replacing $\dot{\mathbf{T}}$ in this equation by its value in equation (35) yields:

$$\mathbf{T} \cdot (\dot{\mathbf{U}}_\mathbf{t} + \mathbf{U_t} \times \boldsymbol{\Omega}) = 0$$

This equation combined with (34) shows that **T** is proportional to the cross-product **W** of $\mathbf{U_t}$ and $\dot{\mathbf{U}}_\mathbf{t} + \mathbf{U_t} \times \boldsymbol{\Omega}$:

$$\mathbf{W} = \mathbf{U_t} \times (\mathbf{U_t} \times \boldsymbol{\Omega} + \dot{\mathbf{U}}_\mathbf{t}) \tag{37}$$

$$\mathbf{T} = \epsilon \frac{\mathbf{W}}{\|\mathbf{W}\|} \tag{38}$$

Where $\epsilon = \pm 1$. We can assume that $\epsilon = 1$ by orienting correctly (c_τ) and (C). Using the derivation rule (18) and noticing that $\partial_{\mathbf{n}_\beta}\mathbf{Om} = \beta \begin{bmatrix} \mathbf{n} \\ 0 \end{bmatrix}$ (or applying the second equation in (12)), we can compute $\dot{\mathbf{U}}_\mathbf{t}$:

$$\dot{\mathbf{U}}_\mathbf{t} = (\kappa w + \frac{\partial \beta}{\partial s})\mathbf{U_n} - \beta \mathbf{t} \times \mathbf{n} \tag{39}$$

where $\mathbf{U_n}$ is equal to $\mathbf{Om} \times \begin{bmatrix} \mathbf{n} \\ 0 \end{bmatrix}$.

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[2]This equation is satisfied by any constant length vector attached to (C).

Equations (37) and (38) are important because they relate in a very simple manner the tangent $\mathbf{T}$ to the unknown 3D curve (C) to the known vector $\mathbf{U_t}$, the angular velocity $\boldsymbol{\Omega}$ and and to $\dot{\mathbf{U}}_\mathbf{t}$. Notice that this last vector contains the unknown tangential real optical flow w.

Furthermore, $\mathbf{W}$ itself satisfies a differential equation. Indeed, we can write:

$$\mathbf{W} = \|\mathbf{W}\| \frac{\mathbf{W}}{\|\mathbf{W}\|}$$

Taking the total derivative with respect to time yields:

$$\dot{\mathbf{W}} = \|\dot{\mathbf{W}}\| \frac{\mathbf{W}}{\|\mathbf{W}\|} + \boldsymbol{\Omega} \times \mathbf{W}$$

since the vector $\frac{\mathbf{W}}{\|\mathbf{W}\|}$ is of unit length, and therefore satisfies equation (35). To compute $\|\dot{\mathbf{W}}\|$, we write:

$$\|\mathbf{W}\| = (\mathbf{W} \cdot \mathbf{W})^{1/2}$$

Therefore:

$$\|\dot{\mathbf{W}}\| = \frac{\dot{\mathbf{W}} \cdot \mathbf{W}}{\|\mathbf{W}\|}$$

The final equation is:

$$(\mathbf{W} \cdot \mathbf{W})\dot{\mathbf{W}} - (\dot{\mathbf{W}} \cdot \mathbf{W})\mathbf{W} + (\mathbf{W} \cdot \mathbf{W})\mathbf{W} \times \boldsymbol{\Omega} = \mathbf{0}$$

Recognizing in the first two terms $\mathbf{W} \times (\dot{\mathbf{W}} \times \mathbf{W})$, we have proved the following theorem:

Theorem 4 *The direction* $\mathbf{W}$ *of the tangent to the 3D curve* (C) *satisfies the following differential equation:*

$$\mathbf{W} \times (\dot{\mathbf{W}} \times \mathbf{W} + (\mathbf{W} \cdot \mathbf{W})\boldsymbol{\Omega}) = \mathbf{0} \tag{40}$$

Equation (40) is fundamental: it expresses the relationship between the unknown geometry and motion of the 3D curve (C) and the geometry and motion of the 2D curve (c_τ).

In order to exploit equation (40), we have to compute $\dot{\mathbf{W}}$. From the definition of $\mathbf{W}$ (equation (37)), we deduce:

$$\dot{\mathbf{W}} = \dot{\mathbf{U}}_\mathbf{t} \times (\mathbf{U_t} \times \boldsymbol{\Omega}) + \mathbf{U_t} \times (\dot{\mathbf{U}}_\mathbf{t} \times \boldsymbol{\Omega} + \mathbf{U_t} \times \dot{\boldsymbol{\Omega}} + \ddot{\mathbf{U}}_\mathbf{t}) \tag{41}$$

Therefore, we also have to evaluate $\ddot{\mathbf{U}}_\mathbf{t}$ (we use again equation (18)):

$$\ddot{\mathbf{U}}_\mathbf{t} = (w^2 \frac{\partial \kappa}{\partial s} + w(\partial_{\mathbf{n}_\beta}\kappa + \frac{\partial^2 \beta}{\partial s^2}) + \partial_{\mathbf{n}_\beta}\frac{\partial \beta}{\partial s} + \kappa \dot{w})\mathbf{U_n} + (\kappa w + \frac{\partial \beta}{\partial s})\dot{\mathbf{U}}_\mathbf{n} - (w\frac{\partial \beta}{\partial s} + \partial_{\mathbf{n}_\beta}\beta)\mathbf{t} \times \mathbf{n}$$

and since:

$$\dot{\mathbf{U}}_\mathbf{n} = -(\kappa w + \frac{\partial \beta}{\partial s})\mathbf{U_t} + w\mathbf{t} \times \mathbf{n} \tag{42}$$

We eventually have:

$$\ddot{\mathbf{U}}_\mathbf{t} = (w^2 \frac{\partial \kappa}{\partial s} + w(\partial_{\mathbf{n}_\beta}\kappa + \frac{\partial^2 \beta}{\partial s^2}) + \partial_{\mathbf{n}_\beta}\frac{\partial \beta}{\partial s} + \kappa \dot{w})\mathbf{U_n} - (\kappa w + \frac{\partial \beta}{\partial s})^2\mathbf{U_t} + (\kappa w^2 - \partial_{\mathbf{n}_\beta}\beta)\mathbf{t} \times \mathbf{n} \tag{43}$$

Notice that, therefore, equation (40) involves w and $\dot{w}$, the real tangential optical flow and its total time derivative, as well as $\dot{\boldsymbol{\Omega}}$, the angular acceleration.

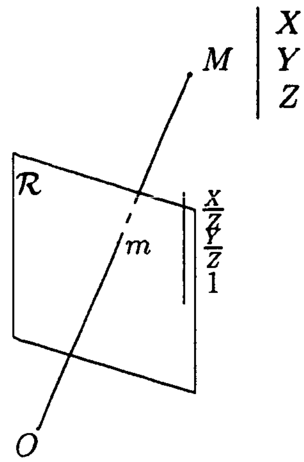

Figure 10: The perspective projection

5.2 Obtaining more equations

We are now going to use the perspective equation (see figure 10):

$$Z\mathbf{Om} = \mathbf{OM} \tag{44}$$

to obtain a number of interesting relations by taking its total time derivative and its derivative with respect to S.

5.2.1 Taking the total time derivative of the perspective equation

Taking the total derivative of equation (44) with respect to time, and applying our derivation rule yields:

$$V_{M_z}\mathbf{Om} + Z(w\begin{bmatrix} \mathbf{t} \\ 0 \end{bmatrix} + \beta\begin{bmatrix} \mathbf{n} \\ 0 \end{bmatrix}) = \mathbf{V}_M \tag{45}$$

where the vector $\mathbf{V}_M$ is the three-dimensional velocity of point M which is of course equal to $\mathbf{V} + \mathbf{\Omega} \times \mathbf{OM}$. Projecting this vector equation onto $\mathbf{t}$ and $\mathbf{n}$ yields two scalar equations:

$$Z(w + \mathbf{\Omega} \cdot \mathbf{b}) = V_t - (\mathbf{Om} \cdot \mathbf{t})V_z \tag{46}$$

$$Z(\beta - \mathbf{\Omega} \cdot \mathbf{a}) = V_n - (\mathbf{Om} \cdot \mathbf{n})V_z \tag{47}$$

where $\mathbf{a}$ and $\mathbf{b}$ are given by:

$$\mathbf{a} = \mathbf{Om} \times \mathbf{U_t} \tag{48}$$

$$\mathbf{b} = \mathbf{Om} \times \mathbf{U_n} \tag{49}$$

These equations are the standard flow equations expressed in our formalism. They are fundamental in the sense that they express the relationship between the unknown 3D motion of a point and its observed 2D motion.

Notice that we can eliminate Z between (46) and (47) and obtain the value of the tangential real optical flow w as a function of $\mathbf{\Omega}$ and $\mathbf{V}$:

$$w = (\beta - \mathbf{\Omega} \cdot \mathbf{a})f(\mathbf{V}) - \mathbf{\Omega} \cdot \mathbf{b} \tag{50}$$

where we have:

$$f(\mathbf{V}) = \frac{V_t - (\mathbf{Om} \cdot \mathbf{t})V_z}{V_n - (\mathbf{Om} \cdot \mathbf{n})V_z} \tag{51}$$

we assume $V_n - (\mathbf{Om} \cdot \mathbf{n})V_z \neq 0$.

5.2.2 Taking the derivative with respect to S of the perspective equation

Taking the partial derivative of equation (44) with respect to S, and applying our derivation rule yields:

$$T_z\mathbf{Om} + uZ \begin{bmatrix} \mathbf{t} \\ 0 \end{bmatrix} = \mathbf{T} \tag{52}$$

T_z is the component of $\mathbf{T}$ along the z-axis. By taking the inner product of both sides with $\mathbf{t}$, and using equation (38), we obtain[3]:

$$uZ\|\mathbf{W}\| = \mathbf{W} \cdot \mathbf{t} - (\mathbf{Om} \cdot \mathbf{t})W_z \tag{53}$$

This equation yields u once the motion $(\mathbf{\Omega}, \mathbf{V})$, and the structure Z have been obtained (see the next section). Note that this allows us to compute the full Frenet frame of (C) by taking derivatives of $\mathbf{T}$ as defined from equation (38) (see appendix B).

5.3 Closing the loop or finding the kinematic screw

The basic idea is to combine equation (40) which embeds the local structure of (C) at M (its tangent) and the fact that it moves rigidly, with equation (50) which is a pure expression of the kinematics of the point M without any reference to the fact that it belongs to a curve.

We take the total time derivative $\dot{w}$ of w, using equation (50). In doing this, we introduce the accelerations $\dot{\mathbf{\Omega}}$ and $\dot{\mathbf{V}}$. If we now replace w and $\dot{w}$ by those values in equation (40), we obtain two polynomial equations in $\mathbf{\Omega}$, $\mathbf{V}$, $\dot{\mathbf{\Omega}}$, and $\dot{\mathbf{V}}$ with coefficients depending on the observed geometry and motion of the 2D curve (the two equations come from the fact that equation (40) is a cross-product). Two such equations are obtained at each point of (c_τ). Those polynomials are of degree 5 in $\mathbf{V}$, 1 in $\dot{\mathbf{V}}$, homogeneous of degree 5 in $(\mathbf{V}, \dot{\mathbf{V}})$, of degree 4 in $\mathbf{\Omega}$, 1 in $\dot{\mathbf{\Omega}}$, and of total degree 9 in all those unknowns.

This step is crucial. This is where we combine the structural information about the geometry of (C) embedded in equation (40) with purely kinematic information about the motion of its points embedded in equation (45). This eliminates the need for the estimation of the real tangential flow w and its time derivative $\dot{w}$. We thus have the following theorem:

Theorem 5 *At each point of (c_τ) we can write two polynomial equations in the coordinates of $\mathbf{\Omega}$, $\mathbf{V}$, $\dot{\mathbf{\Omega}}$ and $\dot{\mathbf{V}}$ with coefficients which are polynomials in quantities that can be measured from the spatio-temporal surface $(\mathcal{S})$:*

$$\begin{array}{ccccc} \beta & \frac{\partial\beta}{\partial s} & \frac{\partial^2\beta}{\partial s^2} & \partial_{\mathbf{n}_\beta}\beta & \partial_{\mathbf{n}_\beta}\frac{\partial\beta}{\partial s} \\ & \kappa & \frac{\partial\kappa}{\partial s} & \partial_{\mathbf{n}_\beta}\kappa & \end{array}$$

Those polynomials are obtained by eliminating w and $\dot{w}$ between equations (41), (50), and (54). They are of total degree 9, homogeneous of degree 5 in $(\mathbf{V}, \dot{\mathbf{V}})$, of degree 5 in $\mathbf{V}$, 1 in $\dot{\mathbf{V}}$, 4 in $\mathbf{\Omega}$, 1 in $\dot{\mathbf{\Omega}}$.

[3] Taking the inner product with $\mathbf{n}$, and replacing $\mathbf{T}$ with $\frac{\mathbf{W}}{\|\mathbf{W}\|}$ is a tautology $0 = 0$

Thus, N points on (c_τ) provide $2N$ equations in the 12 unknowns $\mathbf{\Omega}$, $\mathbf{V}$, $\dot{\mathbf{\Omega}}$, and $\dot{\mathbf{V}}$. Therefore, we should expect to be able to find, in some cases, a finite number of solutions. Degenerate cases where such solutions do not exist can be easily found: straight lines, for example [FDN89], are notorious for being degenerate from that standpoint. The problem of studying the cases of degeneracy is left for further research. Ignoring for the moment those difficulties (but not underestimating them), we can state one major conjecture/result:

Conjecture 1 *The kinematic screw $\mathbf{\Omega}$, $\mathbf{V}$, and its time derivative $\dot{\mathbf{\Omega}}$, $\dot{\mathbf{V}}$, of a rigidly moving 3D curve can, in general, be estimated from the observation of the spatio-temporal surface generated by its retinal image, by solving a system of polynomial equations. Depth can then be recovered at each point through equation (47). The tangent to the curve can be recovered at each point through equation (37).*

Notice that we never actually compute the tangential real optical flow w. It is just used as an intermediate unknown and eliminated as quickly as possible, as irrelevant. Of course, if needed, it can be recovered afterwards, from equation (46).

For completeness, we show how to compute $\dot{w}$. From equation (50), and using equation (18), we can write:

$$\dot{w} = (w\frac{\partial \beta}{\partial s} + \partial_{\mathbf{n}_\beta}\beta - \dot{\mathbf{\Omega}} \cdot \mathbf{a} - \mathbf{\Omega} \cdot \dot{\mathbf{a}}) f(\mathbf{V}) + (\beta - \mathbf{\Omega} \cdot \mathbf{a}) \dot{f}(\mathbf{V}) - \dot{\mathbf{\Omega}} \cdot \mathbf{b} - \mathbf{\Omega} \cdot \dot{\mathbf{b}} \tag{54}$$

From equations (48) and (49), we deduce

$$\dot{\mathbf{a}} = (w\mathbf{t} + \beta\mathbf{n}) \times \mathbf{U_t} + \mathbf{Om} \times \dot{\mathbf{U}}_\mathbf{t}$$

and:

$$\dot{\mathbf{b}} = (w\mathbf{t} + \beta\mathbf{n}) \times \mathbf{U_n} + \mathbf{Om} \times \dot{\mathbf{U}}_\mathbf{n}$$

We have already computed $\dot{\mathbf{U}}_\mathbf{t}$ and $\dot{\mathbf{U}}_\mathbf{n}$ (equations (39) and (42)), therefore:

$$\dot{\mathbf{a}} = (w\mathbf{t} + \beta\mathbf{n}) \times \mathbf{U_t} + (\kappa w + \frac{\partial \beta}{\partial s})\mathbf{b} - \beta \mathbf{Om} \times (\mathbf{t} \times \mathbf{n}) \tag{55}$$

$$\dot{\mathbf{b}} = (w\mathbf{t} + \beta\mathbf{n}) \times \mathbf{U_n} - (\kappa w + \frac{\partial \beta}{\partial s})\mathbf{a} + w \mathbf{Om} \times (\mathbf{t} \times \mathbf{n}) \tag{56}$$

Finally, we have to compute $\dot{f}(\mathbf{V})$; from its definition (equation (51)), we deduce:

$$\begin{gathered} \dot{f}(\mathbf{V})(V_n - (\mathbf{Om} \cdot \mathbf{n})V_z)^2 = \\ (\dot{V}_t + (\kappa w + \tfrac{\partial \beta}{\partial s})V_n - (w + (\kappa w + \tfrac{\partial \beta}{\partial s})(\mathbf{Om} \cdot \mathbf{n}))V_z - (\mathbf{Om} \cdot \mathbf{t})\dot{V}_z)(V_n - (\mathbf{Om} \cdot \mathbf{n})V_z) - \\ (V_t - (\mathbf{Om} \cdot \mathbf{t})V_z)(\dot{V}_n - (\kappa w + \tfrac{\partial \beta}{\partial s})V_t - (\beta - (\kappa w + \tfrac{\partial \beta}{\partial s})(\mathbf{Om} \cdot \mathbf{t}))V_z - (\mathbf{Om} \cdot \mathbf{n})\dot{V}_z) \end{gathered} \tag{57}$$

In this expression, we have $\dot{V}_t = \dot{\mathbf{V}} \cdot \mathbf{t}$, etc...

5.4 Some practical considerations

From theorem 5, it appears that in order to compute the coefficients of the two polynomial equations, the following quantities have to be evaluated:

$$\begin{array}{ccccc} \beta & \frac{\partial \beta}{\partial s} & \frac{\partial^2 \beta}{\partial s^2} & \partial_{\mathbf{n}_\beta}\beta & \partial_{\mathbf{n}_\beta}\frac{\partial \beta}{\partial s} \\ & \kappa & \frac{\partial \kappa}{\partial s} & \partial_{\mathbf{n}_\beta}\kappa & \end{array}$$

Notice that, because of equation (31), the term $\partial_{\mathbf{n}_\beta}\kappa$ does not need to be estimated. I have not yet investigated the practical problems related to the evaluation of these quantities, but from the theoretical viewpoint, there are no difficulties, as long as the spatio-temporal surface $(\mathcal{S})$ can be observed. Some considerations related to this are developed in section 7.

6 Example of retinal motion

This is a simple example, but a useful one in practice, where the motion of (C) is in the retina plane (or in a plane parallel to it). In this case, the apparent and real optical flows are identical. If d the distance of the plane to the retina, the motion of m is described by

$$\mathbf{v}_m = \mathbf{v}_a = \mathbf{v}_r = \frac{\mathbf{V}}{d} + \mathbf{\Omega} \times \mathbf{Om}$$

where $\mathbf{V}$ is parallel to the retina plane, and $\mathbf{\Omega}$ is perpendicular to it. This implies that

$$\alpha = w = \frac{\mathbf{V}}{d} \cdot \mathbf{t} + \mathbf{\Omega} \cdot \mathbf{U_t} \tag{58}$$

and:

$$\beta = \frac{\mathbf{V}}{d} \cdot \mathbf{n} + \mathbf{\Omega} \cdot \mathbf{U_n} \tag{59}$$

Differentiating equation (58) with respect to s, we readily obtain:

$$\frac{\partial \alpha}{\partial s} = \frac{\partial w}{\partial s} = \kappa(\frac{\mathbf{V}}{d} \cdot \mathbf{n} + \mathbf{\Omega} \cdot \mathbf{U_n}) = \kappa\beta$$

which is of course equation (27).

This is an interesting result that says that for a rigid planar motion, the derivative of the tangential velocity of a point on the curve with respect to arc-length is equal to the product of the curvature with the normal velocity at that point.

The simplicity of this case allows us to verify that the condition $\partial_{\mathbf{n}_\beta}\beta = \kappa(1+\beta^2)$ is not satisfied. Using the second equation of (18), we write:

$$\dot{\beta} = w\frac{\partial \beta}{\partial s} + \partial_{\mathbf{n}_\beta}\beta$$

We now compute $\dot{\beta}$ and $\partial_{\mathbf{n}_\beta}\beta$ from equation (59); after some algebra, it turns out that (we omit d for simplicity):

$$\dot{\beta} = \dot{\mathbf{V}} \cdot \mathbf{n} + \dot{\mathbf{\Omega}} \cdot \mathbf{U_n} - (\kappa w + \frac{\partial \beta}{\partial s})\mathbf{V} \cdot \mathbf{t} + \Omega \cdot \dot{\mathbf{U}}_\mathbf{n}$$

Using equation (42) for $\dot{\mathbf{U}}_\mathbf{n}$ and equation (58):

$$\dot{\beta} = \dot{\mathbf{V}} \cdot \mathbf{n} + \dot{\mathbf{\Omega}} \cdot \mathbf{U_n} - w(\kappa w + \frac{\partial \beta}{\partial s}) + w\mathbf{\Omega} \cdot (\mathbf{t} \wedge \mathbf{n})$$

We also compute $\frac{\partial \beta}{\partial s}$:

$$\frac{\partial \beta}{\partial s} = -\kappa w + \mathbf{\Omega} \cdot (\mathbf{t} \wedge \mathbf{n})$$

Replacing this expression in the expression of $\dot{\beta}$ yields:

$$\dot{\beta} = \dot{\mathbf{V}} \cdot \mathbf{n} + \dot{\mathbf{\Omega}} \cdot \mathbf{U_n}$$

Using equation (18) we obtain:

$$\partial_{\mathbf{n}_\beta}\beta = \dot{\mathbf{V}} \cdot \mathbf{n} + \dot{\mathbf{\Omega}} \cdot \mathbf{U_n} + \kappa w^2 - w\mathbf{\Omega} \cdot (\mathbf{t} \wedge \mathbf{n})$$

We thus notice that $\partial_{\mathbf{n}_\beta}\beta$ involves the accelerations $\dot{\mathbf{V}}$ and $\dot{\mathbf{\Omega}}$ whereas the term $\kappa(1+\beta^2)$ involves only velocities. Therefore, in general, $\partial_{\mathbf{n}_\beta}\beta \neq \kappa(1+\beta^2)$, which confirms our analysis of section 4.4.

7 Discussion of implementation issues

There are at least two ways of implementing this theory. The first is to pursue the approach of the spatio-temporal surface, i.e to extract pieces of curves from images, approximate the corresponding spatio-temporal surface, and solve the polynomial equations without going back to the original intensities. This might be called the *symbolic* approach. The second approach is to go back to the original intensity image and see how the various quantities that we need to estimate can be described in terms of the spatio-temporal image and its space and time derivatives. This might be called the *signal* approach.

I now sketch the way those two approaches might be implemented.

7.1 The symbolic approach

We first describe a possible way of approximating the spatio-temporal surface $(\mathcal{S})$ with quadric patches and then how to estimate the various derivatives which are needed in equation (40).

7.1.1 Approximating $(\mathcal{S})$

In practice, $(\mathcal{S})$ is defined by a set of curves (c_τ) observed in the image at times $\tau_0, \tau_1, \cdots, \tau_n$. Given a point m on (c_τ), we want to approximate $(\mathcal{S})$ in the neighbourhood of the corresponding point P. A standard way of doing this is to fit an analytical surface, for example a quadric $(\mathcal{Q})$, to the points of $(\mathcal{S})$ in the vicinity of P. Such a quadric has an equation:

$$\frac{1}{2}\mathbf{R}^T\mathbf{A}\mathbf{R} + \mathbf{R}^T\mathbf{b} + c = 0$$

where $\mathbf{R} = [x, y, \tau]^T$, $\mathbf{A}$ is a 3×3 symmetric matrix, $\mathbf{b}$ a 3×1 vector, and c a real number. The quadric $(\mathcal{Q})$ is represented by the 10 numbers defining $\mathbf{A}$, $\mathbf{b}$, and c, which we represent as the parameter vector $\mathbf{q}$.

Techniques for approximating surfaces with quadrics have been developed previously [FH86, SZ90]. For our present purposes, we minimize with respect to $\mathbf{q}$ the following criterion:

$$\sum_{P_i \text{ in a neighborhood of } P} \left(\frac{1}{2}\mathbf{P}_i^T\mathbf{A}\mathbf{P}_i + \mathbf{P}_i^T\mathbf{b} + c\right)^2$$

subject to the following two linear constraints:

$$\frac{1}{2}\mathbf{P}^T\mathbf{A}\mathbf{P} + \mathbf{P}^T\mathbf{b} + c = 0 \tag{60}$$

$$\mathbf{P}^T\mathbf{A}\begin{bmatrix} \mathbf{t} \\ 0 \end{bmatrix} + \mathbf{b}^T\begin{bmatrix} \mathbf{t} \\ 0 \end{bmatrix} = 0 \tag{61}$$

The constraint (60) means that the point P of interest is on the quadric $(\mathcal{Q})$, and the constraint (61) means that the vector $[\mathbf{t}, 0]^T$ is in the tangent plane to $(\mathcal{Q})$ at P. The reason for this will become clearer in the next section. Since the criterion to be minimized and the constraints, are homogeneous functions of $\mathbf{q}$, in order to avoid the trivial solution $\mathbf{q} = \mathbf{0}$, we must add another constraint. In our previous work [FH86], we succesfully used the constraint:

$$Tr(\mathbf{A}\mathbf{A}^T) = 1 \tag{62}$$

which has the advantage of being invariant with respect to rigid transformations. The details for performing the minimization are left to the reader.

From the best quadric defined by $\hat{\mathbf{A}}$, $\hat{\mathbf{b}}$ and $\hat{c}$, we can estimate the normal $\mathbf{N}_P$, the gaussian and mean curvatures H and K, and the principal directions. The normal is given by:

$$\mathbf{N}_P = \hat{\mathbf{A}}\mathbf{P} + \hat{\mathbf{b}} \tag{63}$$

The computation of H, K, and the principal directions are left to the reader.

7.1.2 Actual computation of $\frac{\partial \beta}{\partial s}$, $\partial_{\mathbf{n}_\beta}\beta$, $\frac{\partial^2 \beta}{\partial s^2}$ and $\partial_{\mathbf{n}_\beta}\frac{\partial \beta}{\partial s}$

From the previous section, we have a local representation of $(\mathcal{S})$ in the vicinity of P by a quadric $(\mathcal{Q})$ defined by its cartesian equation:

$$\frac{1}{2}\mathbf{R}^T\hat{\mathbf{A}}\mathbf{R} + \mathbf{R}^T\hat{\mathbf{b}} + \hat{c} = 0 \tag{64}$$

The estimated normal optical flow $\hat{\beta}_P$ at P is defined from the normal $\mathbf{N}_P$ to $(\mathcal{Q})$: $\mathbf{N}_P$ can be written as $[N_{2P}\mathbf{n}^T, N_{3P}]^T$, where $\mathbf{n}$ is the unit vector normal to (c_τ) (because of equation (61)) and $\hat{\beta}_P$ is equal to $-\frac{N_{3P}}{N_{2P}}$. In fact, we can use this formula to define β in the neighborhood of P:

$$\beta = -\frac{N_3}{N_2}$$

From equation (64), the normal $\mathbf{N}$ to $(\mathcal{Q})$ at point R is given by $\mathbf{N} = \hat{\mathbf{A}}\mathbf{R} + \hat{\mathbf{b}}$; this implies that we can express β as a function of x, y and τ. Therefore, we can compute $D\beta = \nabla\beta$, the spatio-temporal gradient, explicitely as a function of x, y, and τ. Similarly, we can compute the coordinates of the vector $[\hat{\beta}_P\mathbf{n}^T, 1]^T$ since $\mathbf{n}$ and $\hat{\beta}_P$ are known. It is nice to notice that, by construction so to speak, the vector $[\hat{\beta}_P\mathbf{n}^T, 1]^T$ is the tangent plane to $(\mathcal{Q})$ since it is easy verify that:

$$\mathbf{N}_P \cdot \begin{bmatrix} \hat{\beta}_P\mathbf{n} \\ 1 \end{bmatrix} = 0$$

Therefore we have:

$$\frac{\partial \beta}{\partial s} = D\beta(\mathbf{t}_0) = \nabla\beta \cdot \mathbf{t}_0$$
$$\partial_{\mathbf{n}_\beta}\beta = \nabla\beta \cdot \mathbf{n}_\beta$$

where in both equations, $\nabla\beta$ is evaluated at P. We also have:

$$\frac{\partial^2 \beta}{\partial s^2} = \nabla(\frac{\partial \beta}{\partial s}) \cdot \mathbf{t}_0$$
$$\partial_{\mathbf{n}_\beta}(\frac{\partial \beta}{\partial s}) = \nabla(\frac{\partial \beta}{\partial s}) \cdot \mathbf{n}_\beta$$

7.1.3 What about κ ?

κ and $\frac{\partial \kappa}{\partial s}$ are estimated directly from the image curve (c_τ). $\partial_{\mathbf{n}_\beta}\kappa$ is computed from equation (31).

7.2 The signal approach

Another interesting possibility is to use the standard approach to optical flow and start with equation (1). It yields an estimate of β, the normal real and apparent optical flow:

$$\beta = -\frac{I_\tau}{\sqrt{I_x^2 + I_y^2}}$$

We can also define κ by analysing the way edges are detected in the image; we use an approach based upon the computation of the image spatial gradient $\mathbf{g} = \begin{bmatrix} I_x \\ I_y \end{bmatrix}$. The curves which are extracted are such that $\mathbf{g}$ is parallel to the unit normal vector $\mathbf{n}$. This yields an expression for $\mathbf{n}$ and $\mathbf{t}$:

$$\mathbf{n} = \frac{1}{g}\begin{bmatrix} I_x \\ I_y \end{bmatrix} \tag{65}$$

$$\mathbf{t} = \frac{1}{g}\begin{bmatrix} -I_y \\ I_x \end{bmatrix} \tag{66}$$

where g is the magnitude of $\mathbf{g}$.

The curves detected therefore satisfy at each point the following relation:

$$\mathbf{t} \cdot \mathbf{g} = 0$$

Differentiating this expression with respect to arclength s yields:

$$\kappa(\mathbf{n} \cdot \mathbf{g}) + \mathbf{t}^T \mathbf{H} \mathbf{t} = 0$$

where $\mathbf{H}$ is the spatial Hessian of the intensity function:

$$\mathbf{H} = \begin{bmatrix} I_{x^2} & I_{xy} \\ I_{xy} & I_{y^2} \end{bmatrix}.$$

This yields the following expression for κ:

$$\kappa = \frac{2I_y I_{yx} I_x - I_{x^2} I_y^2 - I_{y^2} I_x^2}{(I_x^2 + I_y^2)^{3/2}} \tag{67}$$

In fact there is a further equation that is satisfied and comes from the way edges are detected in practice: in standard implementations of this gradient approach [Can86], only pixels such that g^2 is maximum in the direction of $\mathbf{g}$ are kept for further processing. Mathematically, this implies that:

$$\partial_{\mathbf{g}}(\mathbf{g} \cdot \mathbf{g}) = 0$$

which is equivalent to:

$$\mathbf{g}^T \mathbf{H} \mathbf{g} = 0$$

and therefore:

$$I_x^2 I_{x^2} + 2 I_x I_y I_{xy} + I_y^2 I_{y^2} = 0 \tag{68}$$

This equation can be used to simplify the expression of κ in (67):

$$\kappa = 4\frac{I_x I_y I_{xy}}{\sqrt{(I_x^2 + I_y^2)}} \tag{69}$$

Having defined β and κ in terms of the intensity function $I(x, y, \tau)$ and its partial derivatives, we have to compute the quantities

$$\begin{array}{cccc} \frac{\partial \beta}{\partial s} & \frac{\partial^2 \beta}{\partial s^2} & \partial_{\mathbf{n}_\beta}\beta & \partial_{\mathbf{n}_\beta}\frac{\partial \beta}{\partial s} \\ & \frac{\partial \kappa}{\partial s} & \partial_{\mathbf{n}_\beta}\kappa & \end{array}$$

This is very easy to do, and in fact quite similar to the procedure followed in the previous symbolic approach. Let us treat the general case of a function f of x, y, τ for which we want to compute $\frac{\partial f}{\partial s}$ and $\partial_{\mathbf{n}_\beta} f$. We just have to remenber that:

$$\frac{\partial f}{\partial s} = \nabla f \cdot \mathbf{t}_0 \tag{70}$$

$$\partial_{\mathbf{n}_\beta} f = \nabla f \cdot \mathbf{n}_\beta \tag{71}$$

where ∇ is to be understood as the spatio-temporal gradient of f:

$$\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_\tau \end{bmatrix}$$

If we apply this to the functions β and κ we obtain formulas involving only the image intensity function and its space and time derivative up to the order 3. Those formulas are a bit lengthy and are given for completeness in the appendix.

8 Conclusion

I have studied the relationship between the 3D motion of a curve (C) moving isometrically and the motion of its image (c_τ). I have introduced the notion of real and apparent optical flows and shown how they can be interpreted in terms of vector fields defined on the spatio-temporal surface $(\mathcal{S})$ generated by (c_τ).

I have completely characterized $(\mathcal{S})$ up to a rigid displacement by its first and second fundamental forms and the Gauss and Codazzi-Mainardi equations.

I have shown that the full apparent flow and the normal real flow can be recovered from the differential properties of that surface, but not the real tangential flow.

I have then shown that if the motion of (C) is rigid, then two polynomial equations in the components of its kinematic screw and its time derivative, with coefficients obtained from geometric properties of the surface $(\mathcal{S})$, can be written for each point of (c_τ). In doing this, the role of the spatio-temporal surface $(\mathcal{S})$ is essential since it is the natural place where all the operations of derivation of the geometric features of the curves (c_τ) take place. Fundamentally, those derivations are Lie derivatives with respect to the apparent and real optical flow fields $\mathbf{V}_a$ and $\mathbf{V}_r$ defined on $(\mathcal{S})$. Conditions under which those equations yield a finite number of solutions have not yet been studied.

I think that the major contribution of this paper is to state what can be computed from the sequence of images, under which assumptions about the observed 3D motions, and how. I also believe that similar ideas can be used to study more general types of motions than rigid ones.

This paper also raises two interesting questions which I believe are worth pursuing:

- what is it exactly that published algorithms on the computation of optical flow compute ? since I have shown that without the rigidity assumption, it is impossible to compute the tangential real flow, my suspicion is that the apparent tangential flow is what is actually computed. My analysis shows that the computation can be reduced to a simple integration through equation (29).
- is the optical flow a useful representation ? since I have shown in the case of a rigid motion that the problem could be posed and solved directly in terms of the 3D motion which, after all, is the quantity we are trying to compute in Computer Vision, it is not at all clear that the tangential real optical flow needs to be computed. This may perhaps have some incidence on the way neurophysiologists look at the problem.

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A Proof that $L_{[\mathbf{n}_\beta,\, \mathbf{t}_0]}\beta = \frac{\partial\alpha}{\partial s}\frac{\partial\beta}{\partial s}$

We use the notations of section (2.4). If $\mathbf{X}(u^1, u^2)$ and $\mathbf{Y}(u^1, u^2)$ are two tangent vector fields, the Lie bracket $[\mathbf{X}, \mathbf{Y}]$ of $\mathbf{X}$ and $\mathbf{Y}$ is defined by its coordinates in the tangent plane T_P by the formulas [Dem79]:

$$[\mathbf{X}, \mathbf{Y}]^i = X^j \frac{\partial Y^i}{\partial u^j} - Y^j \frac{\partial X^i}{\partial u^j} \tag{72}$$

where we have used the Einstein convention.

In order to apply this, we express $\mathbf{n}_\beta$ and $\mathbf{t}_0$ in the coordinate system defined by $\mathbf{P}'_S$ and $\mathbf{P}'_\tau$ (we have $u^1 = S$ and $u^2 = \tau$):

$$\mathbf{t}_0 = \frac{1}{u}\mathbf{P}'_S \tag{73}$$

$$\mathbf{n}_\beta = -\frac{w}{u}\mathbf{P}'_S + \mathbf{P}'_\tau \tag{74}$$

It is now easy to verify, using equations (72) that:

$$[\mathbf{n}_\beta,\, \mathbf{t}_0]^1 = \frac{1}{u^2}(w_S - u_\tau)$$

and:

$$[\mathbf{n}_\beta,\, \mathbf{t}_0]^2 = 0$$

Using the facts that $w = v + \alpha$, $v_S = u_\tau$, and the first of the derivation rules (12), we have finally:

$$[\mathbf{n}_\beta,\, \mathbf{t}_0]^1 = \frac{1}{u}\frac{\partial\alpha}{\partial s}$$

and this implies that:

$$L_{[\mathbf{n}_\beta,\, \mathbf{t}_0]}\beta = \frac{\partial\alpha}{\partial s}\frac{\partial\beta}{\partial s}$$

B Computing the Frenet frame of (C)

We rewrite equation (38) as $\|\mathbf{W}\|\mathbf{T} = \mathbf{W}$ and take its derivative with respect to S; using the second of the three-dimensional Frenet formulas (3), we have:

$$u\frac{\partial\|\mathbf{W}\|}{\partial s}\mathbf{T} + \frac{\|\mathbf{W}\|}{R}\mathbf{N} = u\frac{\partial \mathbf{W}}{\partial s} \tag{75}$$

where R is the radius of curvature of (C). Since $\|\mathbf{W}\| = (\mathbf{W}\cdot\mathbf{W})^{1/2}$, we have:

$$\frac{\partial\|\mathbf{W}\|}{\partial s} = \frac{\mathbf{W}\cdot\frac{\partial \mathbf{W}}{\partial s}}{\|\mathbf{W}\|}$$

Replacing in equation (75) and doing some algebra yields:

$$\frac{\mathbf{N}}{R} = \frac{u}{(\mathbf{W}\cdot\mathbf{W})^{3/2}}\mathbf{W}\wedge(\frac{\partial \mathbf{W}}{\partial s}\wedge\mathbf{W})$$

This yields the value of the radius of curvature R as:

$$R = \frac{(\mathbf{W}\cdot\mathbf{W})^{3/2}}{u\|\mathbf{W}\wedge(\frac{\partial \mathbf{W}}{\partial s}\wedge\mathbf{W})\|} \tag{76}$$

where we can replace u by its value from equation (53); it also yields the direction $\mathbf{X}$ of the normal $\mathbf{N}$ as:

$$\mathbf{X} = \mathbf{W}\wedge(\frac{\partial \mathbf{W}}{\partial s}\wedge\mathbf{W}) \tag{77}$$

The binormal $\mathbf{B}$ is equal to $\mathbf{T}\wedge\mathbf{N}$ and the torsion ρ can be obtained by deriving the equation $\|\mathbf{X}\|\mathbf{N} = \mathbf{X}$ with respect to S and applying the third of the three-dimensional Frenet formulas (3):

$$u\frac{\mathbf{X}\cdot\frac{\partial \mathbf{X}}{\partial s}}{\|\mathbf{X}\|}\mathbf{N} - \|\mathbf{X}\|(\frac{\mathbf{T}}{R} + \rho\mathbf{B}) = u\frac{\partial \mathbf{X}}{\partial s}$$

After some algebra and using equation (38), we obtain:

$$\rho\mathbf{B} = \frac{u}{\|\mathbf{X}\|^3}\mathbf{X}\wedge(\mathbf{X}\wedge\frac{\partial \mathbf{X}}{\partial s}) - \frac{\mathbf{W}}{R\|\mathbf{W}\|} \tag{78}$$

which yields ρ when we use equation (53) for u and equation (76) for R.

C A few formulas

We give here the formulas expressing $\frac{\partial\beta}{\partial s}$, $\frac{\partial^2\beta}{\partial s^2}$, $\partial_{\mathbf{n}_\beta}\beta$, $\partial_{\mathbf{n}_\beta}\frac{\partial\beta}{\partial s}$, $\frac{\partial\kappa}{\partial s}$, $\partial_{\mathbf{n}_\beta}\kappa$ as functions of the intensity function and its space and time derivatives; those formulas have been obtained using the symbolic computation system MAPLE.

$$\frac{\partial\beta}{\partial s} = \frac{I_{x\tau}I_x^2I_y + I_{x\tau}I_y^3 - I_\tau I_{x^2}I_xI_y - I_\tau I_{yx}I_y^2 - I_{y\tau}I_x^3 - I_{y\tau}I_y^2I_x + I_\tau I_{yx}I_x^2 + I_\tau I_{y^2}I_yI_x}{(I_x^2+I_y^2)^2}$$

$$\begin{aligned}\partial_{\mathbf{n}_\beta}\beta = (&2I_\tau I_x^3I_{x\tau} + 2I_\tau I_xI_{x\tau}I_y^2 - I_\tau^2I_{x^2}I_x^2 - 2I_\tau^2I_{yx}I_yI_x\\ &+2I_\tau I_yI_{y\tau}I_x^2 + 2I_\tau I_y^3I_{y\tau} - I_\tau^2I_{y^2}I_y^2 - I_{\tau^2}I_x^4\\ &-2I_{\tau^2}I_y^2I_x^2 - I_{\tau^2}I_y^4)/(I_x^2+I_y^2)^{5/2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\beta}{\partial s^2} = (&-I_{x^2}I_{y\tau}I_x^2I_y^3 - I_{x^2\tau}I_y^6 + I_\tau I_{x^3}I_x^3I_y^2 + I_\tau I_{x^3}I_xI_y^4\\ &-I_{y^2\tau}I_x^6 - 2I_\tau I_{y^2x}I_y^4I_x - I_\tau I_{y^2}I_y^4I_{x^2} + 3I_{x\tau}I_{x^2}I_x^3I_y^2\\ &+3I_{x\tau}I_{x^2}I_xI_y^4 + I_\tau I_{yx^2}I_y^5 - 2I_{y^2\tau}I_x^4I_y^2 - I_{y^2\tau}I_y^4I_x^2\\ &-I_{x^2\tau}I_x^4I_y^2 - 2I_{x^2\tau}I_x^2I_y^4 - 4I_{y\tau}I_y^4I_{yx}I_x - I_\tau I_{x^2}I_x^4I_{y^2}\\ &-2I_{yx}I_{y\tau}I_x^3I_y^2 - I_\tau I_{y^2x}I_x^3I_y^2 + 3I_{y\tau}I_{y^2}I_yI_x^4 + 3I_{y\tau}I_{y^2}I_y^3I_x^2\\ &+I_\tau I_{y^3}I_yI_x^4 + I_\tau I_{y^3}I_y^3I_x^2 - 2I_{y^2}I_{x\tau}I_y^4I_x - 2I_{yx}I_{x\tau}I_x^2I_y^3\\ &-4I_{yx}I_{x\tau}I_x^4I_y - 2I_{x^2}I_{y\tau}I_x^4I_y - I_\tau I_{yx^2}I_x^2I_y^3 - 2I_\tau I_{yx^2}I_x^4I_y\\ &-3I_\tau I_{x^2}^2I_x^2I_y^2 + 6I_\tau I_{yx}I_y^3I_{y^2}I_x + 6I_\tau I_{x^2}I_x^2I_{y^2}I_y^2 + 12I_\tau I_{yx}^2I_y^2I_x^2\\ &-10I_\tau I_{x^2}I_xI_{yx}I_y^3 + 6I_\tau I_{x^2}I_x^3I_{yx}I_y - 10I_\tau I_{yx}I_x^3I_{y^2}I_y - 3I_\tau I_{y^2}^2I_y^2I_x^2\\ &+I_\tau I_{x^2}^2I_y^4 + 2I_{yx}I_{y\tau}I_x^5 + I_\tau I_{y^2x}I_x^5 + I_\tau I_{y^2}^2I_x^4\\ &+2I_{x\tau}I_y^5I_{yx} + I_{x\tau}I_x^5I_{y^2} + 4I_{yx\tau}I_x^3I_y^3 + 2I_{yx\tau}I_y^5I_x\\ &+2I_{yx\tau}I_x^5I_y + I_{y\tau}I_y^5I_{x^2} - 2I_\tau I_{yx}^2I_y^4 - 2I_\tau I_{yx}^2I_x^4 - I_{y^2}I_{x\tau}I_y^2I_x^3)/(I_x^2+I_y^2)^{7/2}\end{aligned}$$

$$\begin{aligned}\partial_{\mathbf{n}_\beta}\frac{\partial\beta}{\partial s} = (&2I_yI_{y\tau}^2I_x^5 + 4I_y^3I_{y\tau}^2I_x^3 + 2I_y^5I_{y\tau}^2I_x - 5I_\tau I_{y^2}I_yI_x^4I_{x\tau}\\ &-10I_\tau I_{yx}I_x^4I_yI_{y\tau} - 3I_\tau I_{y^2}I_y^3I_x^2I_{x\tau} - 5I_\tau I_{y^2}I_y^2I_{y\tau}I_x^3 - 6I_\tau I_{y^2}I_y^4I_{y\tau}I_x\\ &-4I_\tau I_{yx}I_x^5I_{x\tau} + 5I_\tau I_{x^2}I_xI_y^4I_{y\tau} + 6I_\tau I_{yx}I_y^2I_x^3I_{x\tau} + 10I_\tau I_{yx}I_y^4I_xI_{x\tau}\\ &-6I_\tau I_{yx}I_y^3I_{y\tau}I_x^2 + 4I_\tau I_{yx}I_y^5I_{y\tau} + 6I_\tau I_{x^2}I_x^4I_{x\tau}I_y + 5I_\tau I_{x^2}I_x^2I_{x\tau}I_y^3\\ &+3I_\tau I_{x^2}I_x^3I_y^2I_{y\tau} + 3I_\tau^2I_{y^2}^2I_y^3I_x - 4I_\tau^2I_{x^2}I_xI_{y^2}I_y^3 - 3I_\tau^2I_{yx}I_y^4I_{y^2}\\ &+3I_\tau^2I_{x^2}I_x^4I_{yx} - 8I_\tau^2I_{yx}^2I_y^3I_x + 8I_\tau^2I_{yx}^2I_yI_x^3 - 3I_\tau^2I_{x^2}^2I_x^3I_y\\ &+12I_\tau^2I_{yx}I_y^2I_{y^2}I_x^2 + 4I_\tau^2I_{x^2}I_x^3I_{y^2}I_y - 12I_\tau^2I_{x^2}I_x^2I_{yx}I_y^2 + 2I_{y^2}I_{x\tau}I_y^5I_\tau\\ &+I_{\tau^2}I_{y^2}I_yI_x^5 - I_\tau^2I_{y^3}I_y^2I_x^3 - I_\tau^2I_{y^3}I_y^4I_x - I_\tau^2I_{yx^2}I_x^5\\ &-2I_{x^2\tau}I_x^5I_yI_\tau - 3I_{y\tau^2}I_x^5I_y^2 - 3I_{y\tau^2}I_x^3I_y^4 - I_{y\tau^2}I_y^6I_x\\ &-I_\tau^2I_{yx}I_{y^2}I_x^4 + I_\tau^2I_{yx^2}I_y^2I_x^3 + 2I_\tau^2I_{yx^2}I_y^4I_x - I_{y\tau^2}I_x^7 + I_{x\tau^2}I_y^7\\ &-I_\tau^2I_{y^2}^2I_x^3I_y - 4I_\tau I_{x^2\tau}I_x^3I_y^3 - 2I_\tau I_{x^2\tau}I_xI_y^5 - 2I_{x\tau}I_y^4I_{y\tau}I_x^2\\ &-2I_{\tau^2}I_{x^2}I_x^3I_y^3 - I_{\tau^2}I_{x^2}I_x^5I_y + I_\tau^2I_{yx}I_{x^2}I_y^4 + 2I_{\tau^2}I_{y^2}I_y^3I_x^3\\ &+I_{\tau^2}I_{y^2}I_y^5I_x - I_{\tau^2}I_{x^2}I_xI_y^5 - I_{\tau^2}I_{yx}I_y^4I_x^2 - I_\tau I_{x^2}I_{x\tau}I_y^5\\ &-2I_\tau I_{yx\tau}I_y^4I_x^2 + 2I_\tau I_{yx\tau}I_x^4I_y^2 + 2I_\tau I_{y^2\tau}I_yI_x^5 + 4I_\tau I_{y^2\tau}I_y^3I_x^3\\ &+2I_\tau I_{y^2\tau}I_y^5I_x + I_\tau I_{y^2}I_{y\tau}I_x^5 + I_{\tau^2}I_{yx}I_x^4I_y^2 + 2I_{y\tau}I_x^4I_{x\tau}I_y^2\\ &+I_\tau^2I_{y^2x}I_y^5 - 2I_{x\tau}I_y^6I_{y\tau} - 4I_x^3I_{x\tau}^2I_y^3 - 2I_x^5I_{x\tau}^2I_y\\ &-I_{\tau^2}I_{yx}I_y^6 - 2I_\tau I_{yx\tau}I_y^6 - 2I_xI_{x\tau}^2I_y^5 + I_{x\tau^2}I_x^6I_y\\ &+3I_{x\tau^2}I_x^4I_y^3 + 3I_{x\tau^2}I_y^5I_x^2 + 2I_\tau I_{yx\tau}I_x^6 + I_{\tau^2}I_{yx}I_x^6\\ &+2I_{y\tau}I_x^6I_{x\tau} + I_\tau^2I_{x^3}I_x^2I_y^3 + I_\tau^2I_{x^3}I_x^4I_y + I_\tau^2I_{x^2}^2I_y^3I_x\\ &-2I_{x^2}I_{y\tau}I_x^5I_\tau - 2I_\tau^2I_{y^2x}I_yI_x^4 - I_\tau^2I_{y^2x}I_y^3I_x^2)/(I_x^2+I_y^2)^4\end{aligned}$$

$$
\begin{aligned}
\tfrac{\partial\kappa}{\partial s} = (&3I_{x^2}I_{y^2}I_xI_y^3 - 9I_{y^2}I_x^2I_{yx}I_y^2 + 3I_{y^2x}I_x^2I_y^3 - 3I_{x^2}^2I_y^3I_x \\
&-3I_{yx}I_{x^2}I_y^4 + 6I_{yx}^2I_xI_y^3 - 6I_{yx}^2I_x^3I_y + 3I_{y^2x}I_x^4I_y \\
&-I_{y^3}I_x^3I_y^2 + 3I_{y^2}^2I_x^3I_y + 3I_{y^2}I_{yx}I_x^4 + I_{x^3}I_y^5 - I_{y^3}I_x^5 \\
&+I_{x^3}I_y^3I_x^2 - 3I_{x^2}I_{y^2}I_x^3I_y + 9I_{yx}I_{x^2}I_y^2I_x^2 - 3I_y^2I_{yx^2}I_x^3 \\
&-3I_y^4I_{yx^2}I_x)/(I_x^2 + I_y^2)^3
\end{aligned}
$$

and:

$$
\begin{aligned}
\partial_{\mathbf{n}_\beta}\kappa = (&-I_{x^2}I_{y\tau}I_x^2I_y^3 - I_{x^2\tau}I_y^6 + I_\tau I_{x^3}I_x^3I_y^2 + I_\tau I_{x^3}I_xI_y^4 \\
&-I_{y^2\tau}I_x^6 - 2I_\tau I_{y^2x}I_y^4I_x - I_\tau I_{y^2}I_y^4I_{x^2} + 3I_{x\tau}I_{x^2}I_x^3I_y^2 \\
&+3I_{x\tau}I_{x^2}I_xI_y^4 + I_\tau I_{yx^2}I_y^5 - 2I_{y^2\tau}I_x^4I_y^2 - I_{y^2\tau}I_y^4I_x^2 \\
&-I_{x^2\tau}I_x^4I_y^2 - 2I_{x^2\tau}I_x^2I_y^4 - 4I_{y\tau}I_y^4I_{yx}I_x - I_\tau I_{x^2}I_x^4I_{y^2} \\
&-2I_{yx}I_{y\tau}I_x^3I_y^2 - I_\tau I_{y^2x}I_x^3I_y^2 + 3I_{y\tau}I_{y^2}I_yI_x^4 + 3I_{y\tau}I_{y^2}I_y^3I_x^2 \\
&+I_\tau I_{y^3}I_yI_x^4 + I_\tau I_{y^3}I_y^3I_x^2 - 2I_{y^2}I_{x\tau}I_y^4I_x - 2I_{yx}I_{x\tau}I_x^2I_y^3 \\
&-4I_{yx}I_{x\tau}I_x^4I_y - 2I_{x^2}I_{y\tau}I_x^4I_y - I_\tau I_{yx^2}I_x^2I_y^3 - 2I_\tau I_{yx^2}I_x^4I_y \\
&-3I_\tau I_{x^2}^2I_x^2I_y^2 + 6I_\tau I_{yx}I_y^3I_{y^2}I_x + 4I_\tau I_{x^2}I_x^2I_{y^2}I_y^2 + 8I_\tau I_{yx}^2I_y^2I_x^2 \\
&-6I_\tau I_{x^2}I_xI_{yx}I_y^3 + 6I_\tau I_{x^2}I_x^3I_{yx}I_y - 6I_\tau I_{yx}I_x^3I_{y^2}I_y \\
&-3I_\tau I_{y^2}^2I_y^2I_x^2 + 2I_{yx}I_{y\tau}I_x^5 + I_\tau I_{y^2x}I_x^5 + 2I_{x\tau}I_y^5I_{yx} \\
&+I_{x\tau}I_x^5I_{y^2} + 4I_{yx\tau}I_x^3I_y^3 + 2I_{yx\tau}I_y^5I_x + 2I_{yx\tau}I_x^5I_y \\
&+I_{y\tau}I_y^5I_{x^2} - 2I_\tau I_{yx}^2I_y^4 - 2I_\tau I_{yx}^2I_x^4 - I_{y^2}I_{x\tau}I_y^2I_x^3)/(I_x^2 + I_y^2)^{7/2}
\end{aligned}
$$

Interestingly enough, with those weak assumptions, it can be verified that equation (31) **holds** true. Note also that those formulas involve at most third order derivatives **of the intensity** function but that the third order time derivative I_{τ^3} never appears.

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